Hypersea Whitepaper (v.1.0 Preview)

Anatoly Ressin[1], Aliaksandr Radyna[2], and Alexander Bokhenek[3]

[1] Blockvis Labs, anatoly@blockvis.com
[2] aliaksandr.radyna@blockvis.com
[3] Byzantine Solutions, ab@byzantine.solutions

Abstract. Decentralized finance is about financial empowerment. Access to open markets, financial products and services, assets with different risk profiles, free of entry barriers, unreasonable restrictions, and artificial gatekeeping. Among other things, decentralized finance is about access to yields: being able to leverage one's capital, however limited, to participate in the global economy. And be fairly rewarded for it.
Hypersea is a novel design of a trustless capital efficient AMM with passive LP management, built on top of an entire new foundational mathematical apparatus for reasoning about CFMM designs and formal analysis of liquidity concentration. This paper conveys the architecture of Hypersea and presents our main mathematical results to date (in Annex, that probably could be published as several separate dedicated papers)

1 Introduction

1.1 On Equitable Access to Exposure

Until AMMs were introduced and popularized, earning from liquidity provision was only possible for highly specialized, well-capitalized parties able to commit lots of time and money. One had to either become such a party, or entrust their capital to one, taking on counterparty risk and paying predatory fees. Automated market maker (AMM) designs of decentralized exchanges (DEXes) allowed anyone to get exposure by becoming a passive liquidity provider (LP). Their capital would go to facilitate liquidity depth on the AMM, improving slippage for traders and collecting fees in exchange.

Active evolution of AMM designs has been ongoing since 2017, and is still being innovated upon. The current iteration of research looks at the question of capital efficiency: how to set up liquidity provision (or the trading function) in a way that offers the best market for the same inventory. Two current groups of contenders are Uniswap v3-like designs (offering the LP to dynamically manage their liquidity) and Curve-like designs (fixing the trading function to a highly concentrated scenario around a specific price point — currently viable only for stablecoin-to-stablecoin pairs).

Hypersea is a novel AMM protocol that offers a next step in that evolution: **automated dynamic liquidity concentration**. We propose a trustless, decentralized, noncustodial protocol that dynamically adjusts liquidity concentration based on performance of observed markets, factoring in levels of uncertainty, potential oracle failures, and issues with specific external markets. Hypersea combines the better properties of modern designs, allowing set-and-forget provision of concentrated liquidity, depeg-resistant, and IL-free.

To build that, we've had to develop a whole new class of mathematical language for AMMs,— discovering some interesting results in the process,— and to produce several gadgets supporting the design. This litepaper tells the story of **Hypersea**.

1.2 Problem Overview and Previous Work

After the first-generation AMM designs [1] — which remain highly relevant to this day — there arose questions of improving protocol efficiency in various ways. Constant-product AMMs generally suffer from two problems: impermanent loss and capital efficiency.

The former boils down to the fact that as an asset gets repriced on external markets (whereby a new price is discovered), AMMs keep selling their LPs' assets at a lower price point, up until arbitrageurs get the AMM up to speed.

The issue of capital efficiency connects to the observation that most trading volume happens in rather narrow price ranges, while liquidity is, in a sense, deployed uniformly over the entire curve, covering ranges where it will never be used.

At a high level, there have been two general approaches to evolving curve-based AMMs.

1. **Static curve deformations**. Pioneered by **Michael Egorov** in his seminal paper, StableSwap. Efficient Mechanism for Stablecoin Liquidity (2019) [2], and implemented in **Curve** protocol. The protocol provides very high concentration of liquidity at a pre-defined price point, offset by very high slippage closer to the tails of the curve. The curve in question has been optimized by hand for single use. StableSwap works well for Stablecoin-to-Stablecoin pairs that never depeg, and isn't applicable to anything else.
   Egorov generalized this construction to arbitrary pairs by proposing a mechanism that would move the concentration point toward an oracle input. This is covered in the paper Automated Market Making with Dynamic Peg (2021) [3].
   Unfortunately, the construction suffers from two serious problems. This design cannot react well to rapidly changing markets, as it has no way to account for changes in volatility, nor an efficient way to track the "fair price" (TWAP oracles are rather slow), so previously highly concentrated curve can miss both crucial parameters - volatility (that affects concentration) and a peg - keeping a highly concentrated curve on a faulty peg that would trigger loss of liquidity at a bad price point.
2. **Fragmented liquidity**. This approach was introduced by **Uniswap v3**[4] and drove several more projects to similar designs. The design retains the

basic constant-product curve of **Uniswap**, but makes one adjustment: instead of deploying liquidity to the $(0, +\infty)$ interval, liquidity providers can pick their own very granular ranges (for instance, a stable-to-ETH pair at a price range $1,500 to $1,750) and only deploy there. The change makes LP positions nonfungible (implemented as NFTs) and introduces significant gas overheads. Most importantly, Uniswap v3 practically eliminates the notion of passive liquidity provision. While it is still possible to provide liquidity across the entire curve, the MEV market of Just-In-Time liquidity destroys the earning opportunity for passive liquidity providers. The jury is still out on whether this is beneficial for the market overall as the market progressively shifts towards professional players highly sophisticated in both market making and MEV wars.

Both groups of designs go for capital efficiency by offering some form of liquidity concentration. Each group makes its own trade-off in serviceable markets and accessibility.

A more recent follower of the *curve deformations* route is **Dodo** [5]. The protocol runs oracle-driven curve deformations, adjusting both reference price and concentration of liquidity on dual-currency pools. The main drawback that Dodo has is opaque methodology for concentration. Namely, private oracles just pass curve parameters to the pool, with no specificity to how these are defined. Because of that, the protocol has to allow many pools with the same trading pair to accommodate for arbitrary oracles: if someone doesn't trust existing (or Dodo's own) oracles, they can launch their own pool and direct their liquidity themselves. That naturally leads to liquidity fragmentation, with many LPs choosing to run their own pool instead of improving liquidity depth on an existing one.

Another notable mention – outside of the context of concentrated liquidity – is **Swaap Finance** [6]. It's an oracle-driven asymmetric design that aims to offer the same "fair" spread as the reference market does. Since oracle inputs can shift pool without inventory changes, inventory is now decoupled from the trading function, which makes it nontrivial to track pool ownership. To preserve information, Swaap introduces a curious technique of recording inventory weights in relative terms, as opposed to fixing them and deriving ownership from the position on the curve.

In a brilliant paper [7] it is shown that the curvature of a trading curve (or a hyper-surface in multi-currency pools) corresponds to the ability of the system to find a fair price while mimicking real market maker behavior. At the same time the name of the corresponding article "When does the tail wag the dog: curvature and markets" implies a criticism on situations when used curves (surfaces) are too rigid (lacks curvature and adaptability), especially for classical constant-product invariant (like Uniswap V2) [1]. Curves should adapt to market rather than the market should adapt to curves. In [8] Angeris and Chitra provide an exhaustive framework to reason about wide class of AMM – Constant Function Market Makers (CFMM).

In summary, the problem the industry has been trying to solve is whether it's possible to build a capital-efficient (non-fragmented, with concentrated liquidity) market that allows passive liquidity provision and does good inventory management without taking on unnecessary losses (such as impermanent loss). As evidenced by by the breakdown above, no decentralized protocol to date has been able to cover every area,— each one choosing to sacrifice either capital efficiency, or passive liquidity management. There's no silver bullet for AMMs.

Or at least there hasn't been.

1.3 Our Contribution

Hypersea offers a universal solution to capital efficiency of AMMs while preserving passive management of liquidity positions. The protocol captures both universal access to exposure and optimizaiton of slippage for the traders.

Using a novel mathematical apparatus developed in-house, we propose a solution that deforms its trading curve to account for volatility, uncertainty, and arbitrary price movements, and shifts both reference price and liquidity concentration in response.

Below is a comparative Venn diagram, listing all of the key properties as in protocols reviewed and in Hypersea (Fig. 1):

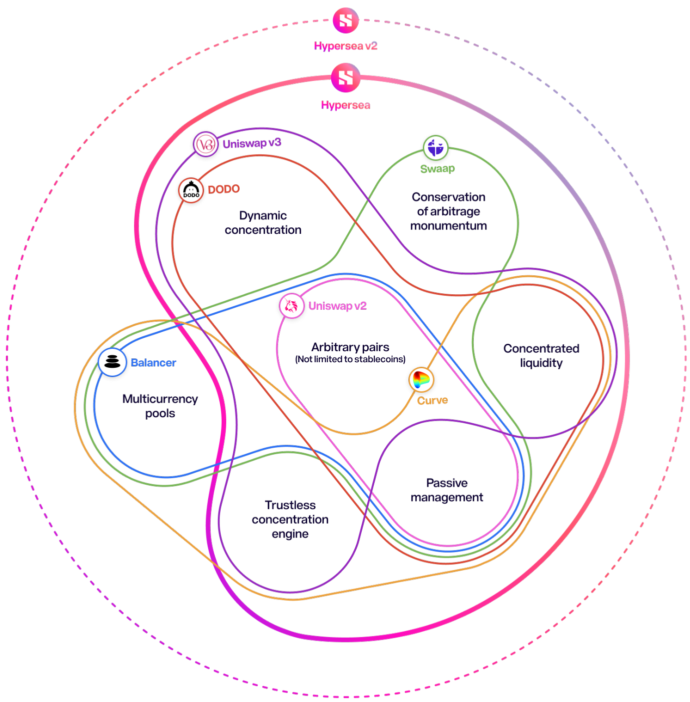

Fig. 1: Venn Diagramm of different AMMs

The actual construction is covered in the next section. Alternative representation of comparison is shown at Fig.2.

	Uniswap v2	Uniswap v3	Curve	Swaap	Balancer	DODO	Hypersea	Hypersea v2
Passive management	✓	✗	✓	✓	✓	✓	✓	✓
Concentrated liquidity	✗	✓	✓	✗	✗	✓	✓	✓
Trustless concentration engine	✗	✓	✓	✗	✗	✗	✓	✓
Dynamic concentration	✗	✓	✗	✗	✗	✓	✓	✓
Conservation of arbitrage momentum	✗	✗	✗	✓	✗	✗	✓	✓
Arbitrary pairs (Not limited to stablecoins)	✓	✓	✗	✓	✓	✓	✓	✓
Multicurrency pools	✗	✗	✓	✓	✓	✗	✗	✓

Fig. 2: Comparative table of different AMMs

2 Hypersea: Automated Market-Maker With Autonomous Liquidity Concentration

2.1 Vision and General Architecture

Hypersea is an AMM that autonomously manages liquidity concentration based on observed market activity. For a volatile market (or under inconsistent market data) concentration will be low, spreading liquidity out to account for bigger expected volatility and protect the LPs' funds from trading at unreliable price points. For a transparent market with low volatility, concentration would be high, providing liquidity depth where it is needed.

All of that is done through curve deformations: expected "fair price" and liquidity concentration both go into the parameters of the curve, deploying funds around the reference target with breadth correlated to volatility and uncertainty. Hypersea uses a range of inputs from both blockchain oracles and on-chain observations, and continuously factors any inconsistencies or flash crashes into liquidity deployment.

Dynamic liquidity concentration enables two unique properties:

1. **Efficient pricing for swappers**: under normal conditions, slippage would be lower than at any other AMM, as the curve would drift towards optimally concentrated liquidity.
2. **Efficient pricing for LPs**: there is virtually no impermanent loss, since the AMM takes specific measures against selling LPs' liquidity below the fair market price (normal slippage aside).

Operationally, the protocol can be thought of as an AMM with static LP positions (set-and-forget), treating all deposited funds in a given market equally. The shape of the curve is adjusted both on oracle updates and from every new trade.

Below is a birds-eye architectural view of Hypersea:

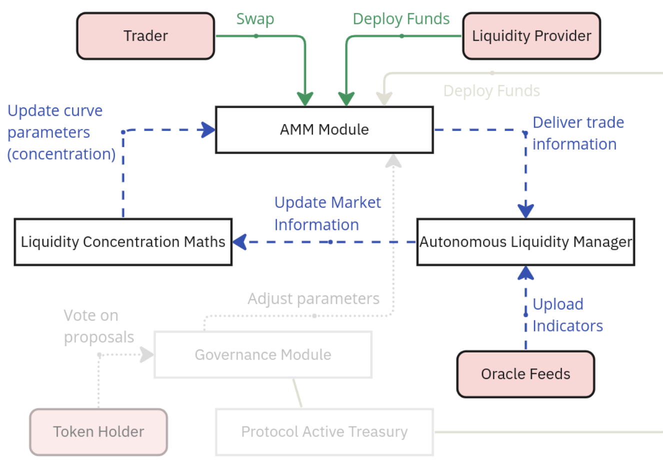

Fig. 3: Birds-eye architectural view of Hypersea

The **Trader** and the **Liquidity Provider (LP)** interact with the protocol like they would with a typical AMM, performing swaps and depositing funds for liquidity. In addition, there are **Oracle Feeds** who upload a range of indicators to the *Autonomous Liquidity Manager* module, alongside updates from the *AMM module* for all swaps performed with Hypersea.

Initially, the protocol will use several of Chainlink's feeds, but the oracle interface is generalized, and there are plans to diversify oracle inputs moving forward. Because of the risk model baked into the *Autonomous Liquidity Manager*, adding oracle feeds (both in terms of data sources and relayers) will directly contribute to the stability of the protocol.

Inputs from the *Autonomous Liquidity Manager* inform how liquidity is deployed to markets.

Before looking at the formulas of liquidity concentration, we need to set the scene by reviewing motivation and introducing the mathematical foundations of Hypersea.

2.2 Correspondence Between Liquidity and Trading Curves

Consider a liquid order book on a centralized exchange. Two typical representations of the market are a sorted list of bids and asks and a chart of market depth on both sides shown at Fig.4

If we took a snapshot of the market and tried to plot final (after trade is executed) price parametrized by size of a market order (in base asset units), we would get a jumpy curve composed of many small linear steps. Each segment

Sum BTC	Amount	Bid	Ask	Amount	Sum BTC
0.01800	0.01800	27832.18695	27862.85100	0.01800	0.01800
0.01977	0.00177	27820.59415	27870.74845	0.00927	0.02727
0.03993	0.02016	27820.59414	27880.39759	0.00949	0.03676
0.08996	0.05003	27816.79511	27880.39760	0.01960	0.05636
0.20410	0.11414	27813.43725	27881.00840	0.01026	0.06662
0.22379	0.01969	27809.20655	27884.97577	0.04776	0.11438
0.37881	0.15502	27809.08218	27886.43877	0.13168	0.24606
0.58846	0.20965	27806.28430	27887.75867	0.11768	0.36374
0.59937	0.01091	27805.89095	27890.54310	0.15399	0.51773
0.64783	0.04846	27805.01239	27901.46881	0.25057	0.76830

Fig. 4: Order book representations

would correspond to a limit order, keeping marginal execution price at level and marginally pulling the average execution price, and more significantly pulling the final price that will be exposed to market participants. Integrating that curve we will obtain corresponding size of the market order in quote asset units. That can be well illustrated by a scheme Fig.5

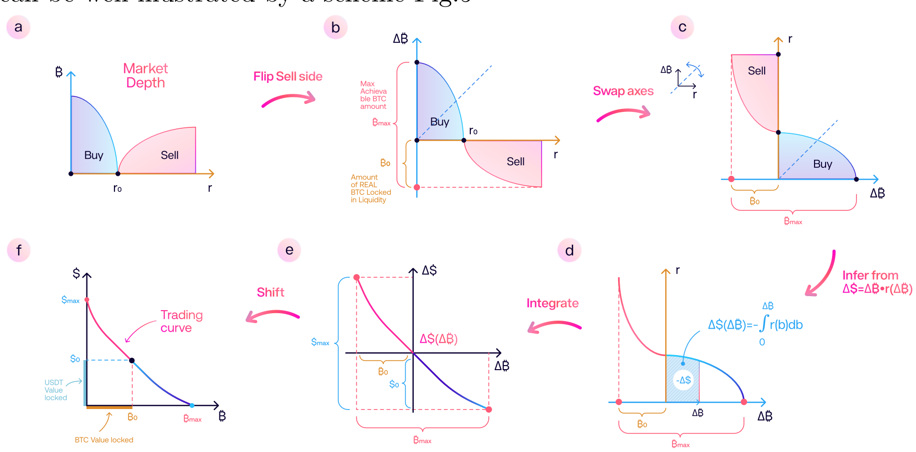

Fig. 5: Six steps in transforming an order book into trading curve

Conversely, if we took any Uniswap market, we could convert it to an order book by approximating the curve by imitating infinitesimal trades.

Once we introduce the notion of **liquidity concentration**, the design space becomes trickier. Intuitively, liquidity concentration is a measure of how hard the trading curve resists price pressure as we move the inventory in one direction along the curve. For Uniswap curve, the resistance is uniform (corresponding to fully deconcentrated liquidity), but it's not immediately apparent. However, it becomes much clearer if we plot what we call a *liquidity concentration chart* – a parametric curve of the (hyperbolic) curvature radius at each price point on the trading curve.

Measuring hyperbolic curvature is based on the following property of the hyperbola: $dy/dx = -Y/X$, this property allows to calculate a spot price just by calculating the ratio between reserves. Geometrically that means that the tangent line at any point of hyperbola is a reflection of the straight line connecting a center of the coordinate system and the point of interest (so together with horizontal axis these elements form an isosceles triangle). That can be visualized by Fig.6.

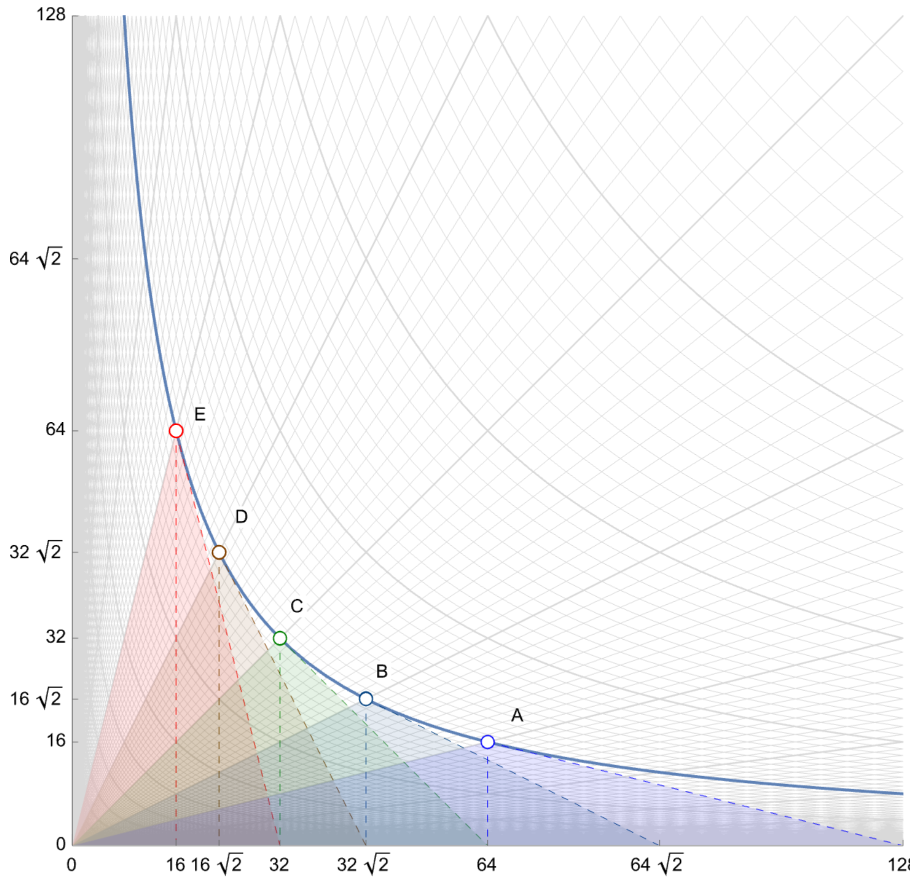

Fig. 6: Isosceles triangles are the key for finding hyperbolic radius, as all reflections of tangents are crossing at the same point (center of hyperbolic coordinates)

For arbitrary trading curve embedded into given orthogonal coordinate system hyperbolic radius at given point could be found using this property. If we think that our curve segment is locally hyperbolic we can just reconstruct a center of local hyperbolic coordinate system using two converging isosceles triangles, and then just to reconstruct a hyperbola and measure its radius as shown on Fig.7

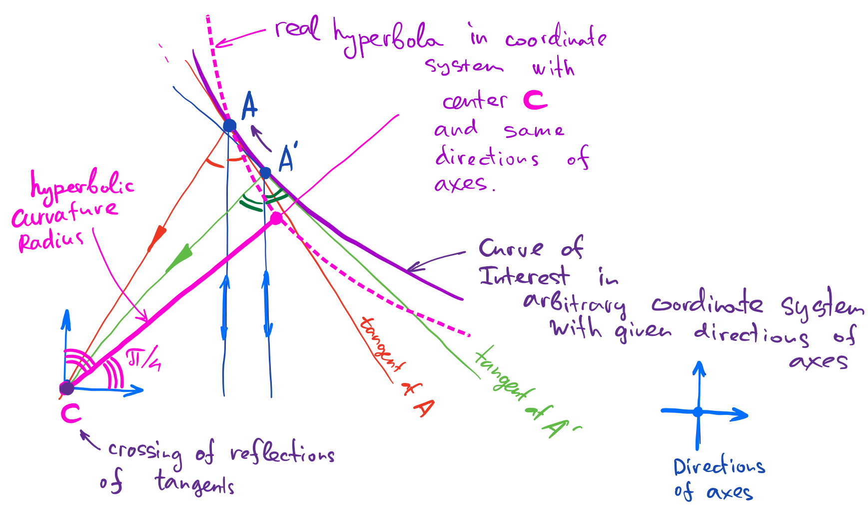

Fig. 7: Measuring hyperbolic curvature radius (as a liquidity concentration measure)

Liquidity concentration is uniform for Uniswap v2, but starts to change as we concentrate liquidity. There is a correspondence between trading curves and liquidity concentration charts, and it is possible to define a formal transformation between the two going in either direction as a generalization of Uniswap V3 piecewise constant liquidity concentration as shown on Fig.9

It is more natural defining liquidity concentration over logarithmic prices rather than on raw prices. So here we will always use $\theta = ln(r)$ where r is raw price (rate). At the left side of Fig.9 there are figures that represent disjoint hyperbolic sectors combinations that corresponds to different liquidity concentration levels. At right side there are corresponding functions that show the dependency beetween hyperbolic radius and corresponding log-price θ.

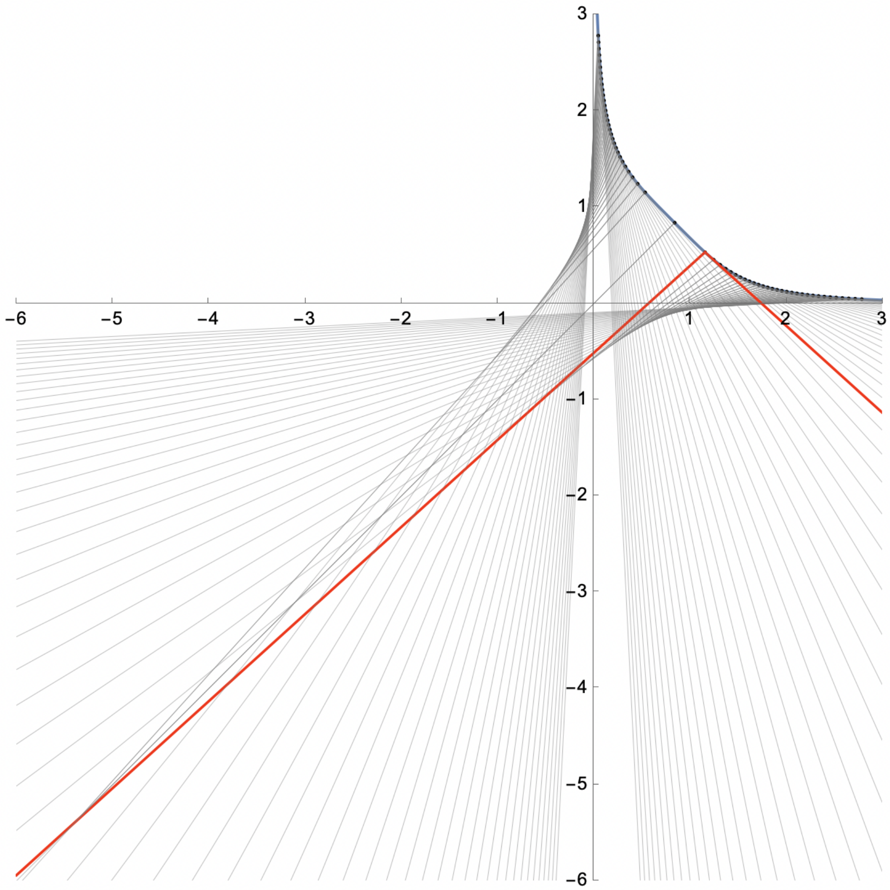

Fig. 8: Example of hyperbolic curvature radius of $x^3y + y^3x = 1$

It is also worth mentioning a visual way how to recombine pieces of hyperbolas with different hyperbolic radii into one single trading curve: see Fig.10.

For further generalization of liquidity concentration we generalize hyperbolic curvature radii to Cobb-Douglas hyperbolic raddii. Just replacing the hyperbola with more general Cobb-Douglas functions (actually isoquants of two-dimentional Cobb-Douglas value-function that is also used in Balancer [9]. So insted of pieces of curves defined by $X \cdot Y = const$ we are moving to more general form $Y^{\omega} \cdot X^{1-\omega} = const$. That allows us (similary to Balancer) for every given positive concentration function and given asset reserves (X, Y) find an appropriate weights ω and $1 - \omega$ that will produce any predefined price r. Visual representation of generalizing trading curves over Cobb-Douglas functions is presented on Fig.11

We found a very elegant parametric form for transforming liquidity concentration function into correspongint trading curve and vice versa. We call the function from liquidity concentration chart to a trading curve Ω-transform:

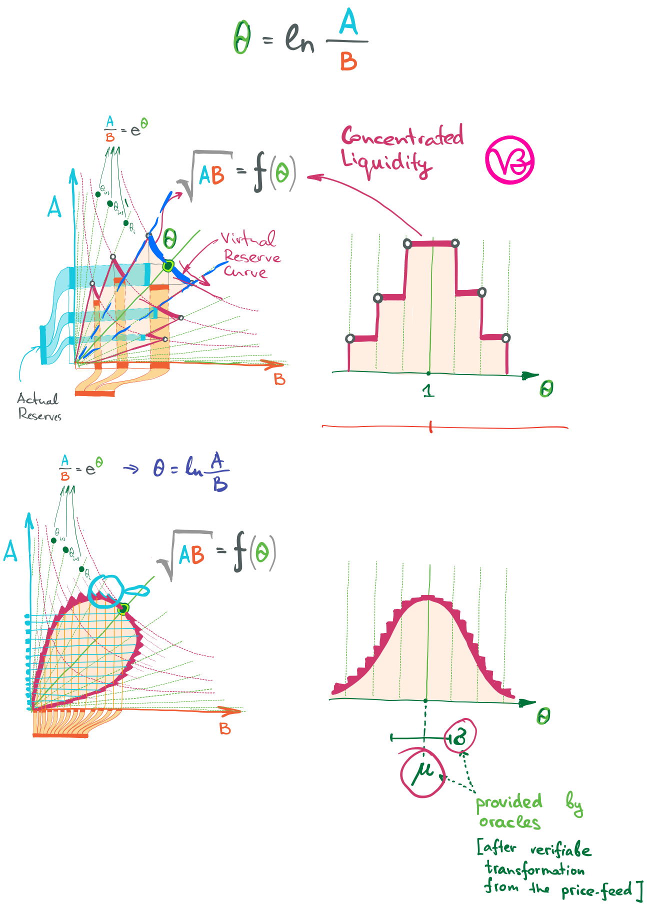

Fig. 9: Generalization of Uniswap piecewise-constant liquidity

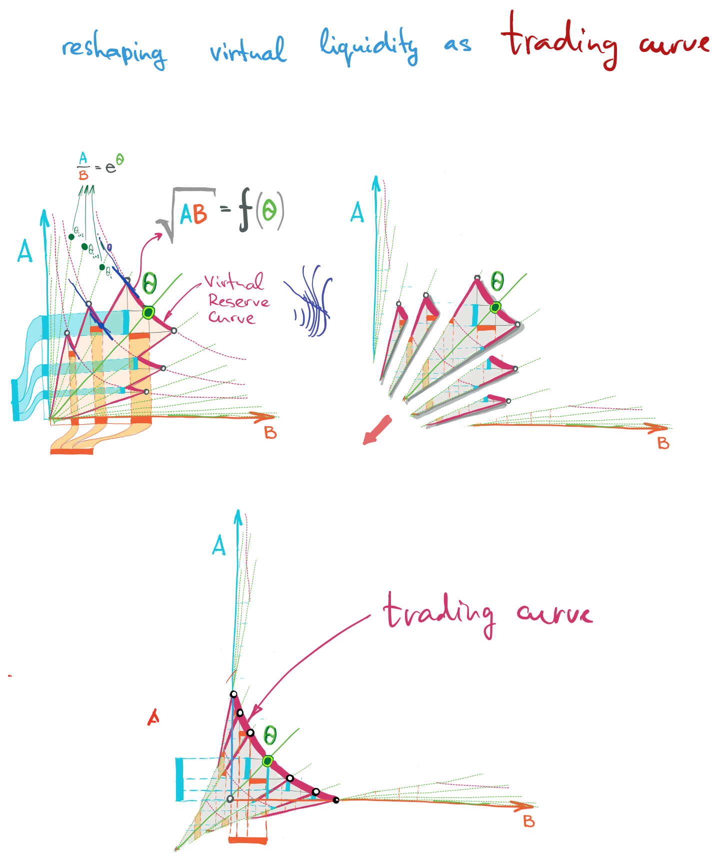

Fig. 10: Combining pieces of different hyperbolas into single trading curve)

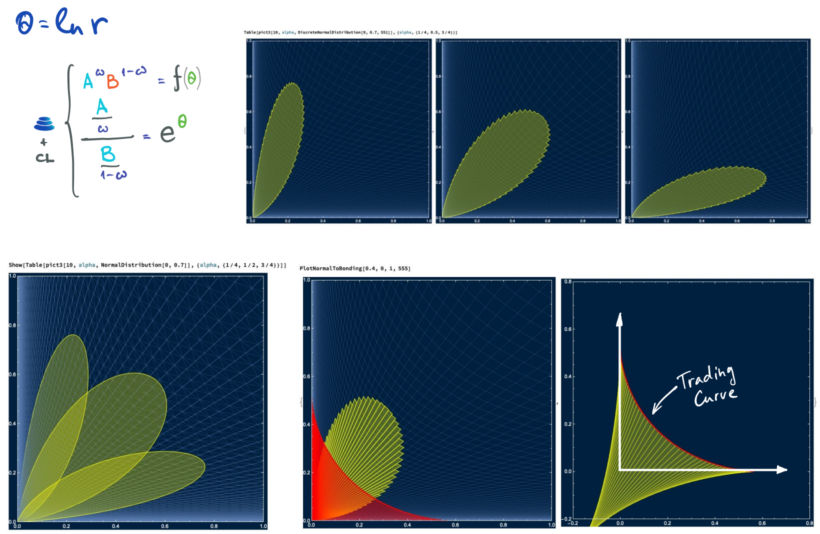

Fig. 11: Generalizing to Cobb-Douglas liquidity concentration.

$$\begin{cases} x(t) = C_\omega \int\limits_{t}^{+\infty} f(\theta) e^{-\omega\theta} d\theta \\ y(t) = C_\omega \int\limits_{-\infty}^{t} f(\theta) e^{(1-\omega)\theta} d\theta \end{cases}, \text{where } C_\omega = \omega(\frac{1-\omega}{\omega})^\omega \tag{1}$$

...and its reverse as *inverse Ω-transform* (in parametric form):

$$\Omega^{-1}(t, \omega) = \langle \log(-F'), \; \frac{(-F')^{(\omega+1)}}{C_\omega \cdot F''} \rangle \tag{2}$$

where $Y = F(X)$ is a trading curve in a closed form.

Hypersea makes a statistical inference about the state of the market, derives from it the desired liquidity concentration, and then runs it through *Ω-transform* to get a trading curve that is then plugged into the AMM.

2.3 Trading Function: CES Curves

Trading function is what determines the parameters of a swap. Given an amount of asset X and choice of the desired asset Y, trading function outputs the amount of asset Y that the swapper would be able to get from the trade. The corresponding curve (or surface, if there are more than two assets) represents acceptable inventories: each trade changes inventory composition (by adding an asset and taking away another asset) and thus moves the inventory along the curve (surface).

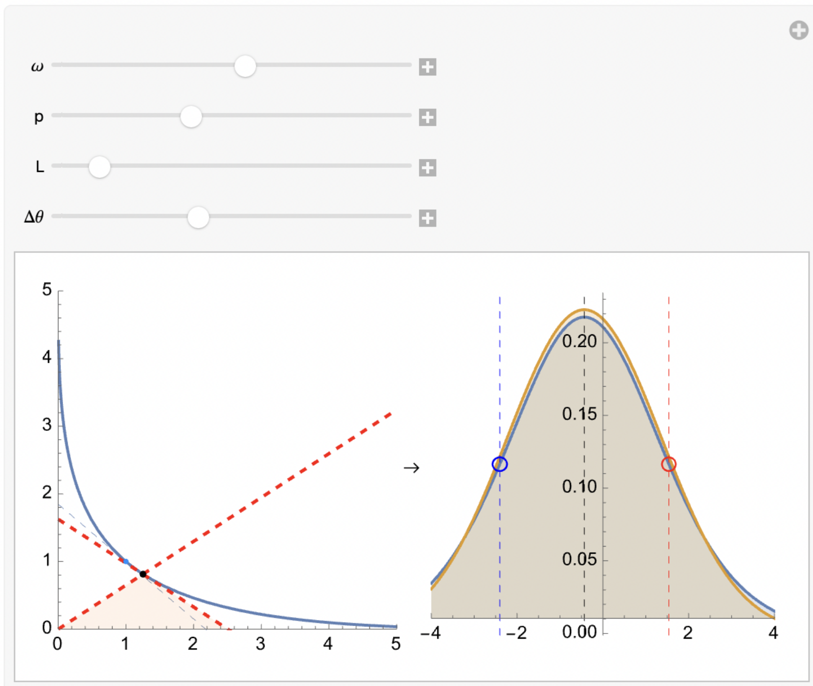

Fig. 12: Gaussian vs CES-liquidity concentration (with same σ)

Curves used by Hypersea come from economic theory of elasticity of substitution. This is not accidental. Conceptually, trading curve on an AMM represents a range of inventory compositions which liquidity providers would be equally happy to hold. This is the deep intuition behind the concept of an automated market maker in general: regardless of the trades made, their size and their amount, LPs should be satisfied with the result. The theory that tracks indifference in substitution came from the early studies of consumer demand in microeconomics. One result in that field has been the family of CES curves (for constant elasticity of substitution), which are interestingly reminiscent of Balancer's weighted pools, however it supports liquidity concentration.

The formula of the curve establishing the relationship between inventories of X and Y is as follows: $\sqrt[p]{\alpha X^p + \beta Y^p} = L$. Note that whenever $p \to 0$, this formula goes to $X^\alpha Y^\beta = L$ (Balancer curve). There is a dependency between α and β, so external parameters are p (curvature) and α or β. Thus, at any moment in time, the trading curve would look like a Balancer curve, but parameters adjust every time concentration has to be changed.

Conveniently, CES curve produces extremely close liquidity concentration to Gaussian distribution (see Fig.12) for which we can efficiently track μ and σ. Moreover (as shown in Annex) we found an exact formula how to infer standard deviation σ of the distribution generated with CES curve liquidity concentration.

That means that we can generate gas-efficient trading curves usable in an AMM. In the following section we will look at ways to target a specific liquidity concentration based on assumptions about the market.

2.4 Distribution-Optimized Liquidity Concentration

Asset price movements are often modelled in finance using variations of Geometric Brownian Motion. The high-level argument for that is that any predictability in movement should be exploitable by market agents in an adversarial environment, leading to some form of wealth redistribution as the inefficiency is eliminated.

Consequently, once a reference price for the given time window is discovered, statistical distribution of trades can be approximated by Gaussian distribution, with mean set to the reference price and variance equal to price volatility in that time window.

Our hypothesis is that optimal concentration of liquidity would also follow the same distribution. Intuitively, this approach allocates funds to price ranges in proportion to the probability of the price reaching them. Thus, deviating from reference price towards the tail with an area of 0.05 would leave 5% of liquidity allocated there. The inverse statement is that we are allocating 95% of liquidity to the range where 95% of the trades are expected to settle.

Starting with a Gaussian distribution of liquidity, the protocol applies Ω-transformation to arrive to a CES cuve ($\sqrt[p]{\alpha X^p + \beta Y^p} = L$), roughly transferring volatility to inverse of the concentration (p) and setting α and β so that the curve passes through the desired inventory (point on the inventory plane, X, Y). As a result of that change, the spot price on the AMM may change from what it was even without any trades taking place on the AMM. This last step is a major difference from all popular AMMs which at this point, essentially, pay arbitrageurs (with LPs' money) to move the price to the reference price. Hypersea does that for free.

One important side note is that the spot price on the AMM does not go directly to the reference price. Instead, it is moved in a way that no new arbitrage opportunities are created. This property is called conservation of arbitrage momentum and is covered in the next section.

2.5 Conservation of Arbitrage Momentum

As trading function adjusts, the protocol should not lose money. This means that whatever arbitrage opportunity was possible before the adjustment, it must be preserved exactly (in terms of extractable profit) after the adjustment.

Let's consider a scenario. The spot price for ETH/WBTC on Hypersea is P_1, and the external reference price is Q_1. Then the reference price changes to Q_2.

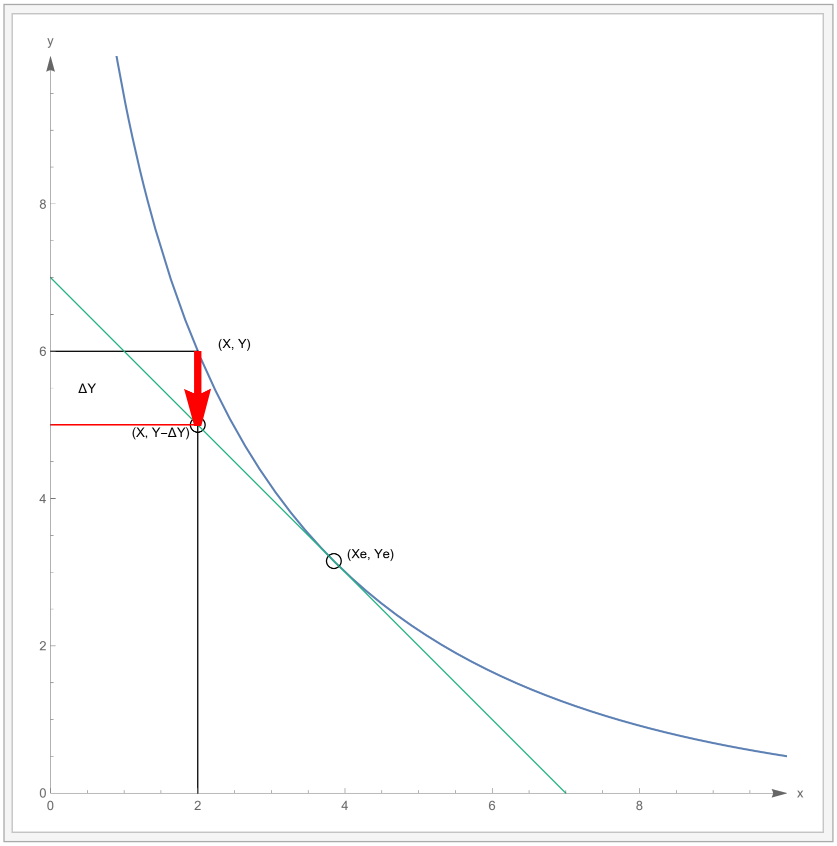

Fig. 13: Arbitrage momentum vs market equilibrium point (Xe,Ye)

If $P_1 = Q_1$, and the AMM just shifts the curve to change the spot price to Q_2, everything is fine. However, if $P_1 \neq Q_1$ and the rule is that we change the spot price to Q_2, an attack is possible:

1. AMM is at equilibrium with external price
2. The attacker makes a swap (at a loss) from asset A to asset B
3. There is an infinitesimal change in the external price
4. The AMM rolls the price back to the external price (as per the hypothetical rule)
5. The attacker now makes the swap opposite to the swap in 2 (selling back whatever liquidity she bought), from B to A

As at step 5 the AMM holds less of asset B than it was at step 2, B is more valuable in terms of A. This means that the attacker would get more of asset A after step 5 than she put in at step 2. Therefore the protocol just lost money (on behalf of its LPs) to the attacker. Since this was assumed as a general rule, the attack is infinitely repeatable — in this exact form, as long as the external price is not too volatile.

The way to counteract this — and the way Hypersea really runs — is to factor the potential arbitrage into the repricing. The law of conservation of arbitrage momentum is that any repricing must preserve the distance (on the trading curve or surface) between the AMM spot price and the reference price.

There is an efficient procedure to arrive to make a correction for arbitrage momentum - see Annex 7.5

It is worthy of note that if the price on the AMM was in sync with the reference price, and then the reference price shifted, the new peg would be equal to the new reference price, as an arbitrage momentum of 0 is being preserved. The real power of this mechanism comes in times of increasing uncertainty: if observed markets start moving around, often there is not enough information to distinguish between fair trading, a market manipulation attempt, an attack on the protocol itself, or an oracle failure. Conservation of arbitrage momentum allows Hypersea to react to new information without committing to new risk.

2.6 Autonomous Decision Making

Changes in liquidity deployment are facilitated by *Autonomous Liquidity Manager* module. It is a decision engine that collects internal and external inputs and informs the AMM on how liquidity should be concentrated. Specific decisions on concentration are made on a per-trade basis and represent a part of trade execution.

The module looks at both internal and external inputs. The goal is to determine three parameters: expected "fair price", asset volatility, and the level of subjective uncertainty. To that effect, Hypersea collects from oracles VWAPs across all relevant markets, infers historical volatility from VWAP records, and compares that with the information about trades on Hypersea itself. Additionally, rate of inventory changes of Hypersea is factored in.

In normal times — when the oracle inputs are consistent, and observed trades don't deviate too far from the variance band,— liquidity concentration is equal, or close to, observed historical volatility. Under inconsistent or volatile conditions,— with conflicting or unavailable oracle feeds, or highly deviant trades,— liquidity starts to deconcentrate, up to the limit of uniform distribution (equivalent to Uniswap v2 curve).

An additional factor that is tracked is relative (logarithmic) inventory changes on Hypersea itself. For instance, if a price feed on one of the stablecoins becomes unavailable, and the notional inventory of that stablecoin started to grow rapidly relevant to other assets,— that could be a soft indication that the stablecoin is depegging, warranting deconcentration of liquidity on the pairs it's traded on.

3 HYPS Token. The Superfluid Protocol Treasury

3.1 Hypersea Governance

The ultimate vision of Hypersea is a fully decentralized autonomous protocol, running its own DAO-driven governance and performing all of the necessary protocol maintenance either autonomously, or driven by its community. To that effect, Hypersea DAO module has full control over every part of the protocol, including contract upgrades, oracle listing and delisting, pool management, and management of the rules applying to the Protocol Active Treasury. The governance module (once it's fully enabled as the protocol is deemed stable) will be driven by the protocol token, HYPS.

3.2 Protocol-Active Treasury and Protocol LP

With dynamic liquidity concentration, Hypersea is in a good position to benefit from a variation of a protocol-owned liquidity design. There is a built in mechanism that connects launch of HYPS token with buildup of the special pool called Protocol Active Treasury (PAT). Liquidity owned by PAT is considered passively provided to all Hypersea pools under two conditions:

1. **No special rules apply**: PAT provides liquidity at the same ratio as current LPs in the pool. That means that in a two-asset pool, liquidity provided by PAT is limited by the asset protocol owns the least of (in USD terms).
2. **Limited liquidity**: PAT will never count as providing more than 50% of liquidity in any given pool. This measure protects protocol-owned liquidity from nontrivial edge attacks with LP manipulation.

The term "passively provided" liquidity stands in this case for the notion that liquidity is not being sent to a specific pool, but is rather considered a Just-in-Time liquidity addition when the trade is made. That means that for the "same" 10 ETH, PAT can provide liquidity on a USDT/ETH trade and on a WBTC/ETH trade right after (assuming it has USDT and WBTC), without moving any of the funds around and staking/unstaking anything.

PAT collects LP fees as a regular LP, which are added back to PAT. Aside from fee collection, the main mechanism for building up PAT is through the initial token offering of HYPS.

3.3 Continuous Token Offering

HYPS token will be offered through a bootstrapping ceremony of the Protocol Active Treasury, taking place over 2 years. The details of the procedure will be published closer to launch date, but its general shape is as follows.

There is a daily mint allocation of HYPS — the maximal amount of tokens that can be bought from the protocol (minted) every day. This amount will be auctioned off using an AMM-run Dutch auction structure. Any unminted tokens will permanently remain unminted, unless at a later date governance will make a decision to the contrary — that will likely depend on the success of PAT as a liquidity source for Hypersea pools.

At this auction, HYPS will be sold for a range of assets, potentially with additional incentives (in form of discounts) depending on the composition of PAT's inventory within a given time frame. All proceeds from selling HYPS tokens will go directly into PAT, become owned by the protocol, and immediately start being deployed to provide liquidity on Hypersea.

3.4 Token Distribution & Allocation

Total supply of HYPS token is 1 billion tokens. Of that, 47% will be minted and distributed over several years through the Continuous Token Offering, and all early stage tokens (Seed, Private, Team, Advisors) will be in lockups in vesting ranging from 12 to 18 months from the token launch event.

4 Conclusion. Markets of the Future

This paper has set the scene for Hypersea, the first AMM with autonomous liquidity concentration.

Arriving to the current design has been a long and adventurous journey, but the future holds so much more. First version of the protocol will feature two-asset pools, similar to most AMMs. However, most of the mathematics we found applies to multidimensional cases as well,— although some problems still remain open.

In the long-term vision, Hypersea could be just one efficient multi-asset pool, not hitting any of the scaling limitations coming from an age of simpler maths. In that world,— one of an open, diverse, equitable market for liquidity provision and asset swaps,— everyone willing would make the best use out of their capital, and worry a bit less about things like active position management.

5 Annex [probably another Paper (or several papers)]

5.1 Overview [probably an Abstract of another Paper(s) :)]

In this work we present a novel type of AMM that is able to concentrate liquidity of bi-currency trading pool in a continuous, automatic and transparent manner based on statistical on-chain inference. We demonstrate that the proposed liquidity concentration engine reacts adequately to various types of non-stationary behaviours of market, reducing divergence losses without sacrificing the efficiency of the provided capital.

Decomposing the trading curve in a given basis. As a foundation we propose a constructive approach for defining a parametrised bijection $\{\Omega_b : \mathcal{C} \to \mathcal{L},\ \Omega_b^{-1} : \mathcal{L} \to \mathcal{C}\}$ (called respectively *direct-* and *inverse Ω-transfrom*) between trading curves $\mathcal{C} : R^+ \to R^+$ and arbitrary liquidity profiles $\mathcal{L} : R \to R_0^+$ within given basis b, where $b \in \mathcal{C}$ is some distinct trading curve called *basis curve*.

We show that if basis b is hyperbolic (specifically $b \equiv \{x \cdot y = 1\}$), then the domain of Ω_b generalises liquidity profiles introduced by Uniswap V3 allowing to use arbitrary non-negative functions instead of only piecewise-constant ones. Meanwhile using $b \equiv \{y^\omega \cdot x^{1-\omega} = 1\}$ where $\omega \in (0; 1)$ allows us to obtain an additional degree of freedom and achieve the flexibility of Balancer in ability to agree the spot price $r = -dy/dx$ on trading curve $c = \Omega_b^{-1}(l)$ at the point of pool reserves $\langle X; Y \rangle$ without introducing unnecessary arbitrage opportunity (and even to control it).

Cobb-Douglas basis and Gaussian Trading Curves We show that basis curves could be reasonably selected as isoquants at unit level of some production function $F(x, y) = 1$, consequently naming basis curves after corresponding production function. Thus the basis $\{y^\omega \cdot x^{1-\omega} = 1\}$ is referred as *Cobb-Douglas basis*. Special attention is paid to Cobb-Douglas basis where we show efficient procedure of changing ω parameter in basis in terms of affecting the trading curve. As well we study general properties of Ω-transform.

We spot unique feature of Gaussian liquidity concentration profiles $N(\mu, \sigma^2)$ in Cobb-Douglas bases: when changing ω they keep their form – Gaussian shape with same variance σ^2 and just change their μ. That means that their trading curve (that we refer as *Gaussian Trading Curve*) is just hyperbolically rotated $(x \mapsto x \cdot k, y \mapsto \frac{y}{k})$. That makes the inverse problem of finding basis $b(\omega)$ effectively solvable just by applying a hyperbolical rotation (e.g. it makes trivial to find a basis in which the spot price at point $\langle X; Y \rangle$ is any predefined value r). The interest in researching Gaussian liquidity profiles is based on easy statistical inference of its parameters ($\mu = EMA, \sigma^2 = EMVAR$) on-chain and heuristic hypothesis standing that *constantly updated Gaussian liquidity profile implements effective Liquidity Provisioning strategy.* We show that ability to resolve a basis allows system to work without external rebalancing the assets.

Gaussian Trading Curves approximation with CES. While Gaussian trading curves are exposing nice properties, it is an open question if they can be computationally effectively implemented on EVM. To address this issue we show that liquidity profiles corresponding to CES curves (*Constant Elastisity Of Substitution*) $\sqrt[p]{\omega \cdot Y^p + (1-\omega) \cdot X^p} = L$ are pretty close to Gaussian liquidity profiles and CES curves can be practically used as approximation of Gaussian trading curves – relative slippage deviation within $[\mu - 3\sigma, \mu + 3\sigma]$ log-price range on the trading curve is less than 1%[1]).

We find an exact formula that makes a relation between parameter p of CES and the standard deviation of its liquidity profile. As well we propose an effective rational approximation that can be easily implemented on EVM. It allows us to transform empirically inferred σ^2 through mixing data from oracles and TWAP-based self-oracle into effective trading curve that will be exposed to traders.

Conservation of Arbitrage Momentum. Another important novel result is a principle of *Conservation of Arbitrage Momentum* that was implicitly used by Swaap.finance yet they did not spotted it as it was just a hidden consequence of their design with no concentration of the liquidity.

We clearly separated that principle and generalized it for any trading curve that is impacted by external information. This principle allows to combine an information produced by internal trades and information produced by extrnal movements of the market in way that no information is lost. We will show that this principle allows to implement Impermanent-Loss-Free architecture for any trading curve including concentrated liquidity.

CES curve management for LP-related risk mitiagtion. We are exploring different versions how to infer a volatility and sudden market movements and translate it to parameters of CES trading curve. Specifically we found the variant when effective σ^2 is just calculated as $\sigma^2 = Max[EMVAR, (EMA - r_{oracle})^2]$ works well in non-stationary conditions of both moderate uncertainty (no daily trend), and sharp situation like stable-coin depegging. While adjusting CES curve's p parameter by the EMA of the asset disproportion log-speed ψ' makes the pool resistant to rapid value losing of the single side in volatile-to-volatile trading pools [Work in Progress].

As recently was proposed by many researchers (like Algebra.finance) dynamic fees based on volatility are applied as well.

That makes Hypersea AMM one of the most risk-mitigating solutions for LP, while demonstrating high capital efficiency.

[1] NB! 1% is not a slippage itself, it is difference between slippages inferred using Gaussian and CES trading curves

5.2 Ω-Transform Formulas Derivation of Trading Curve for Liquidity Concentration Satisfying Cobb-Douglas Invariant Equation

Theorem 1. *For any liquidity concentration $f \in L^1[a;b]$ satisfying Cobb-Douglas invariant* (4) *there is an integral parametric representation of the corresponding Trading Curve:*

$$\begin{cases} x = C_\omega \cdot \int\limits_t^{+\infty} f(\theta) e^{-\omega\theta} d\theta \\ y = C_\omega \cdot \int\limits_{-\infty}^{t} f(\theta) e^{(1-\omega)\theta} d\theta \ , \end{cases} \quad \text{where } C_\omega = \omega \left(\frac{1-\omega}{\omega} \right)^\omega \tag{3}$$

Proof. Let us consider function

$$I_{\theta_i}(\theta) = \begin{cases} 1 & , \text{ if } \theta \in [\theta_i; \ \theta_{i+1}) \\ 0 & , \text{ if } \theta \notin [\theta_i; \ \theta_{i+1}) \end{cases} .$$

Any locally constant function having a finite values $L_0, L_1, \ldots, L_{n-1}$ can be presented as

$$f(\theta) = \sum_{k=0}^{n-1} L_k I_{\theta_k}(\theta) \ .$$

The function has a sense of liquidity concentration and in a general case can be integrable on a segment of real numbers but beforehand we will consider f as a piecewise constant function for the sake of simplicity. Let ω is a fixed number in $(0; 1)$ having a sense of percentage of resource y in a pool of two resources x and y. The following conditions describing Cobb-Douglas invariant equation

$$\begin{cases} y^\omega x^{1-\omega} = f(\theta) \\ \theta = \ln \frac{y}{x} + \Delta_\omega, \end{cases} \text{where } \Delta_\omega = \ln \frac{1-\omega}{\omega}, \text{ and } \Delta_{1/2} = 0 \ . \tag{4}$$

implies that

$$\begin{cases} x = f(\theta) \cdot e^{-\omega\theta} \cdot \left(\frac{1-\omega}{\omega}\right)^\omega \\ y = f(\theta) \cdot e^{(1-\omega)\theta} \cdot \left(\frac{1-\omega}{\omega}\right)^{\omega-1} . \end{cases} \tag{5}$$

We will use the fact that

$$g(t) - g(-\infty) = \int\limits_{-\infty}^{t} g'(\theta) d\theta \tag{6}$$

is a continuous function when g' is a piecewise continuous one with finite jumps.

Let us set up $f_n(\theta) = \sum_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta)$ into the right hand sides of (5).

$$\begin{cases} x = \left(\frac{1-\omega}{\omega}\right)^{\omega} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot e^{-\omega\theta} \\ y = \left(\frac{1-\omega}{\omega}\right)^{\omega-1} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot e^{(1-\omega)\theta} \ . \end{cases} \tag{7}$$

Then we get two piecewise exponent functions which can be glued by the integrals of a kind (6).

$$\begin{cases} x = -\left(\frac{1-\omega}{\omega}\right)^{\omega} \int\limits_{t}^{+\infty} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot (e^{-\omega\theta})' d\theta \\ y = \left(\frac{1-\omega}{\omega}\right)^{\omega-1} \int\limits_{-\infty}^{t} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot (e^{(1-\omega)\theta})' d\theta \ . \end{cases} \tag{8}$$

$$\begin{cases} x = \omega \left(\frac{1-\omega}{\omega}\right)^{\omega} \int\limits_{t}^{+\infty} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot e^{-\omega\theta} d\theta \\ y = (1-\omega) \left(\frac{1-\omega}{\omega}\right)^{\omega-1} \int\limits_{-\infty}^{t} \sum\limits_{k=1}^{n-1} L_k \cdot I_{\theta_k}(\theta) \cdot e^{(1-\omega)\theta} d\theta \ . \end{cases} \tag{9}$$

Formulas (7), (8), (9) can be illustrated by following graphs. We start from Liquidity concentration function given on a log-price domain in reals see 14. Then this function can be transformed in piecewise Cobb-Douglas curves. Each

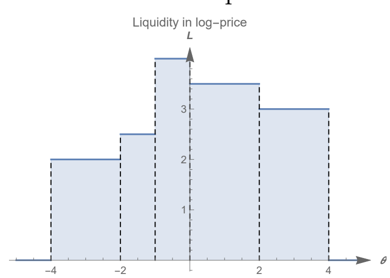

Fig. 14: Liquidity concentration

piece corresponds to a definite constant liquidity concentration L_k see Fig15. Finally, we may glue these pieces by applying $\omega-$Transform to the initial liquidity concentration function f. Since any integrable function can be uniformly

approximated by piecewise constant function above result can be prolongated onto all integrable functions. For any $f \in L^1[a,b]$ there is a piecewise constant function f_n such that

$$\lim_{n\to+\infty} \sup_{t\in[a,b]} |f_n(\theta) - f(\theta)| = 0 \ .$$

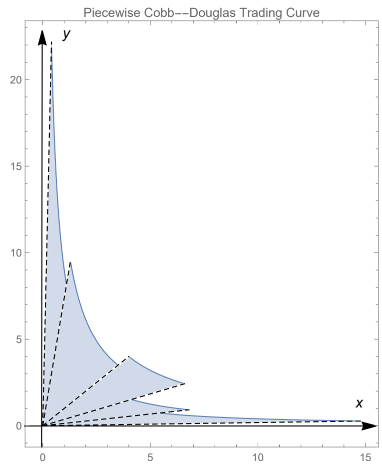

Fig. 15: Piecewise Cobb-Douglas curve

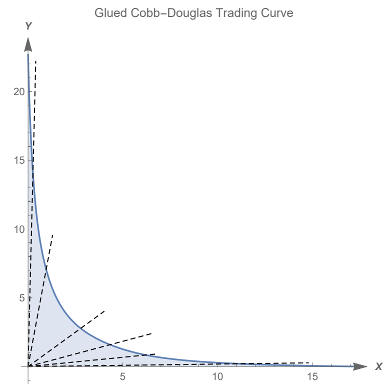

Fig. 16: Glued Piecewise Cobb-Douglas Curve

Therefore we can change f_n in the formula (9) with f because of a uniform convergence of f_n.

We are finally get

$$\begin{cases} x = \omega \left(\frac{1-\omega}{\omega}\right)^{\omega} \cdot \int\limits_{t}^{+\infty} f(\theta) e^{-\omega\theta} d\theta \\ \\ y = (1-\omega) \left(\frac{1-\omega}{\omega}\right)^{\omega-1} \cdot \int\limits_{-\infty}^{t} f(\theta) e^{(1-\omega)\theta} d\theta \ , \end{cases} \tag{10}$$

that is equivalent to (3). This transformation of a function f we call ω-transform and a graph of the result of this transform in a plane $x\ O\ y$ is called the Trading Curve.

We see that direct omega transform of an integrable function f has a form

$$\begin{cases} x = C_\omega \int\limits_{t}^{+\infty} f(\theta) e^{-\omega\theta} d\theta \\ \\ y = C_\omega \int\limits_{-\infty}^{t} f(\theta) e^{(1-\omega)\theta} d\theta \ , \end{cases} \quad \text{where } C_\omega = \omega \left(\frac{1-\omega}{\omega}\right)^{\omega} \tag{11}$$

Now we derive an inverse omega transform by extracting $f(t)$ from (11).

Definition 1 *Denote by $\Omega(f, \omega) : f \mapsto F$ the omega transform with symbol ω. Let F be an image of f. From* (11) *follows that $F(x) = y(x)$.*

Differentiating equations of (11) with respect to t we can get

$$\begin{cases} x'_t = -C_\omega f(t) e^{-\omega t} \\ \\ y'_t = C_\omega f(t) e^{(1-\omega)t} \ , \end{cases} \quad C_\omega = \omega \left(\frac{1-\omega}{\omega}\right)^{\omega}$$

Dividing the second equation onto the first one and applying the chain rule we will get

$$F'_x = \frac{y'_t}{x'_t} = -e^t \ . \tag{12}$$

It is equivalent to

$$t = \log(-F'_x). \tag{13}$$

Let us take a second derivative of F with respect to x:

$$\begin{aligned} F''_{x^2} = (F'_x)'_x = (-e^{t(x)})'_x = -e^{t(x)} \cdot \frac{1}{x'_t(t(x))} \\ = \frac{-e^t}{-C_\omega f(t) e^{-\omega t}} = \frac{e^{(\omega+1)t}}{C_\omega \cdot f(t)} \ . \end{aligned} \tag{14}$$

The original f can be derived from (14) as

$$f(t) = \frac{e^{(\omega+1)t}}{C_\omega \cdot F''_{x^2}(t)} \ . \tag{15}$$

Finally, from (13) and (15) one can get **the inverse omega transform** in a parametric form

$$\left(\log(-F'), \ \frac{(-F')^{(\omega+1)}}{C_\omega \cdot F''} \right) \ . \tag{16}$$

Therefore, we can formulate
Definition 2 *Denote by* $\Omega^{-1}(F, \omega)$ *the inverse omega transform which can be defined by* (16)*. An image of the inverse omega transform is in a space of the originals of the direct omega transform. But the domain of the inverse has to be very delicate.* F *has to be two times differential and can not be a linear or piecewise linear function since* F'' *is in the denominator of* (16) *and can not be equal to zero. We suppose that all trade curves (including Cobb-Douglas curves, CES-curves) are decreasing and strictly convex.*

5.3 Example

Let's consider the following piecewise-defined function as a liquidity profile:

$$f(\theta) = \begin{cases} \sqrt{4 - (\theta + 2)^2} & -4 < \theta \le -2 \\ \sin(\theta\pi) + 2 & -2 < \theta \le 0 \\ 2 & 0 < \theta \le 2 \\ 4 - \theta & 2 < \theta \le 4 \\ 0 & \text{else} \end{cases} \tag{17}$$

Let's apply a sequence of straight and inverse Omega-transformations (with $\omega = 1/2$) to that liquidity profile. On the Fig. 17 you can see, that the liquidity profile was near-perfectly reconstructed.

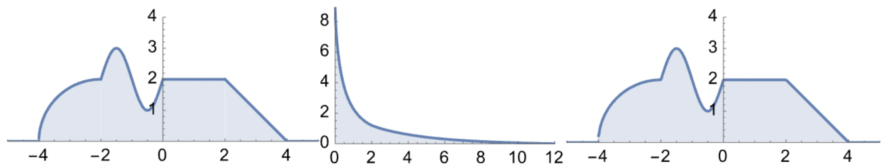

Fig. 17: From left to right: Original liquidity profile $f(\theta)$, Constructed trading curve $F = \Omega(\mathrm{f})$, reconstructed liquidity profile $f = \Omega^{-1}(F)$

6 Direct ω_1 and Inverse ω_2 transform composition $\omega_1 \neq \omega_2$

Recall that $\Omega(f, \omega)$ is an omega transform of a function f with parameter $\omega \in (0,\ 1)$ (see definition 5.2), implemented by formula (3). An image of the function f under Ω we were denoted by F. If we apply Ω^{-1} to F with the same symbol ω we get again f. Formula $f = \Omega^{-1}(\Omega(f, \omega), \omega)$ can be considered as a representation of f in the ω-basis. A reader may compare this formula with well-known Fourier series expansion formula

$$f(t) = \sum_{n \in \mathbf{Z}} c_n e^{2\pi i n t}\ , c_n = \int_0^1 f(x) e^{-2\pi i n x} dx\ ,$$

which is a composition of the direct and the inverse Fourier transforms. In this way it can be risen a question what is going on if someone applies to f a direct omega transform with symbol ω_1 but recover it with the inverse omega transform with symbol ω_2? Due to analogy with Fourier expansion it will corresponds to a changing of the basis. Let us take $\Omega^{-1}(\Omega(f, \omega_1), \omega_2)$. We get

$$\Omega^{-1}(\Omega(f, \omega_1), \omega_2) = \frac{(-F')^{(\omega_2+1)}}{C_{\omega_2} \cdot F''} \tag{18}$$

$$= \frac{e^{(\omega_2+1)t}}{C_{\omega_2} \dfrac{e^t}{C_{\omega_1} f(t) e^{-\omega_1 t}}} = \frac{C_{\omega_1}}{C_{\omega_2}} e^{(\omega_2 - \omega_1)t} f(t)\ . \tag{19}$$

So, the changing of the ω_1-basis onto ω_2-basis for any integrable function f is distinguished from f on the multiplicative exponent.

Note. *If the liquidity concentration* $f \in \mathcal{N}(0; \sigma^2)$ *(i.e.* $f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}}$*) then*

$$\Omega^{-1}(\Omega(f, \omega_1), \omega_2)(t) = \frac{C_{\omega_1}}{C_{\omega_2}} \cdot e^{\frac{(\omega_1-\omega_2)^2}{2\sigma^2}} \cdot f(t - \sigma^2(\omega_2 - \omega_1))\ . \tag{20}$$

Since inverse omega transform for symbol $\omega = 0.5$ is more simple and corresponds to hyperbolic case of the trading curve, we can put in (19) $\omega_2 = 0.5$ and get

$$\Omega^{-1}(\Omega(f,\omega_1),0.5) = C_{\omega_1} e^{(0.5-\omega_1)t} f(t) \ . \tag{21}$$

Eventually, we may rewrite (21) as

$$\Omega(f,\omega_1) = \Omega(C_{\omega_1} e^{(0.5-\omega_1)t} f(t), 0.5) \ . \tag{22}$$

It means ω_1-transform for any $\omega_1 \in (0;\ 1)$ can be expressed via 0.5-transform.

6.1 Example: combining $\Omega_{1/2}$ and $\Omega^{-1}_{1/9}$

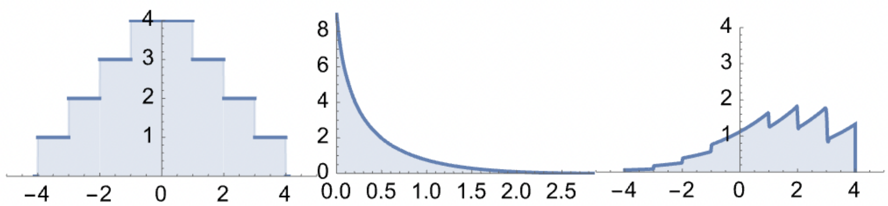

Fig. 18: From left to right: Original liquidity profile $f(\theta)$, Constructed trading curve $F = \Omega_{1/2}$(f), reconstructed liquidity profile $f = \Omega^{-1}_{1/9}(F)$

As we see, the initial liquidity profile is multiplied by exponentially growing factor.

7 Log-price Parameterization of some Trading Curves

7.1 Hyperbola curve parameterization

$$\begin{cases} x(\theta) = L \cdot e^{-\frac{\theta}{2}} \\ y(\theta) = L \cdot e^{\frac{\theta}{2}} \end{cases} , \theta \in (-\infty, +\infty). \tag{23}$$

If we eliminate parameter θ from (23) by multiplying first equation onto second one we will get implicit hyperbola equation

$$x \cdot y = L^2 \ . \tag{24}$$

7.2 Cobb-Douglas Trading Curve Parameterization

$$\begin{cases} x(\theta) = L \cdot e^{-\omega\theta} \\ y(\theta) = L \cdot e^{(1-\omega)\theta} \end{cases} , \theta \in (-\infty, +\infty). \tag{25}$$

If we eliminate parameter θ from (25) by applying power $1-\omega$ to both sides of the fist equation and taking power ω to both sides of the second equation and then multiply first equation onto second one we will get implicit Cobb-Douglas equation

$$x^{1-\omega} \cdot y^{\omega} = L \ . \tag{26}$$

It is clear that if we put $\omega = 0.5$ into (25) and (26) than we get (23) and (24) i.e. hyperbola curve.

Here we need to note very important property of omega transform. It concerns to rule when we toggle from ω_1-basis to ω_2-basis.

7.3 Constant Elasticity of Substitution (CES) Trading Curve Parameterization

Consider a following system which define a two dimensional curve

$$\begin{cases} x(\theta) = \dfrac{L \cdot \left(\frac{\omega}{1-\omega} e^{\theta}\right)^{\frac{1}{1-p}}}{\left(\omega + (1-\omega)\left(\frac{\omega}{1-\omega} e^{\theta}\right)^{\frac{p}{1-p}}\right)^{\frac{1}{p}}} \\ \\ y(\theta) = \dfrac{L}{\left(\omega + (1-\omega)\left(\frac{\omega}{1-\omega} e^{\theta}\right)^{\frac{p}{1-p}}\right)^{\frac{1}{p}}} \end{cases} , \ \theta \in (-\infty, +\infty). \tag{27}$$

Let us eliminate parameter θ from (27) by rising left and right part of both equation to power p, then multiplying the resulting first equation onto $(1-\omega)$ and second one on ω and finally by summing the resulting equations. All these operations together will lead us to implicit CES equation (details see for instance here [10][2]).

$$(1-\omega)x^p + \omega y^p = L^p \ , \tag{28}$$

which is an isoquant of so called utility (or productivity) function

$$U(x, y) = ((1-\omega)x^p + \omega y^p)^{\frac{1}{p}} \ . \tag{29}$$

When put $p = 1$ in (28) we get a linear curve that means full substitution reserve x with y. The opposite case when $p = 0$ will give us independence of reserve x from y. If we take $p = 0$ in (28) we get identity which is useless. Instead more convenient to take exponent $\frac{1}{p}$ from both sides of (28) and then take a limit while $p \to 0$:

$$\lim_{p \to 0} ((1-\omega)x^p + \omega y^p)^{\frac{1}{p}} = L \ . \tag{30}$$

$$\lim_{p \to 0} \ln ((1-\omega)x^p + \omega y^p)^{\frac{1}{p}} = \ln L \ , \tag{31}$$

$$\lim_{p \to 0} \frac{\ln ((1-\omega)x^p + \omega y^p)}{p} = \ln L \ , \tag{32}$$

[2] The article is not available online, but the reader can get all needed information from wikipedia

To overcome uncertainty in left hand side of (32) we will use l'Hopital rule:

$$\lim_{p\to 0} \frac{\left(\ln\left((1-\omega)x^p+\omega y^p\right)\right)'}{(p)'} = \ln L \ , \tag{33}$$

$$\lim_{p\to 0} \frac{\left((1-\omega)x^p+\omega y^p\right)'}{(1-\omega)x^p+\omega y^p} = \ln L \ , \tag{34}$$

$$\lim_{p\to 0} \frac{(1-\omega)x^p\ln x+\omega y^p\ln y}{(1-\omega)x^p+\omega y^p} = \ln L \ , \tag{35}$$

$$(1-\omega)\ln x+\omega\ln y = \ln L \ , \tag{36}$$

which is equivalent to

$$x^{1-\omega}\cdot y^{\omega} = L \ . \tag{37}$$

That is Cobb–Douglas equation, please compare with (26).

7.4 The Inverse Omega Transform for CES Curves

The utility function (29) has widely used in economics. Parameter $p \in [0;1]$ serves as an indicator of elasticity $E(x,y) = \frac{1}{1-p}$ between two resources. For $p = 0$ we get an inelastic case of the utility function

$$U(x,\ y) = A\ x^{1-\omega}\ y^{\omega} \ , \tag{38}$$

which is referred to Cobb–Douglas curve and our standard omega transform have been introduced above, and for $p = 1$ we get an elastic one which is referred to infinite elasticity of substitution or perfect substitution.

$$U(x,\ y) = (1-\omega)\ x+\omega\ y \ , \tag{39}$$

We start from (27) and evaluate derivatives x'_θ, y'_θ.

$$\begin{cases} x'(\theta) = \dfrac{L\cdot\omega(\frac{\omega}{1-\omega}e^{\theta})^{\frac{1}{p-1}}}{(p-1)\cdot\left(\omega+(1-\omega)\cdot\left(\frac{\omega}{1-\omega}e^{\theta}\right)^{\frac{p}{p-1}}\right)^{1+\frac{1}{p}}} \\ \\ y'(\theta) = -\dfrac{L\cdot(1-\omega)(\frac{\omega}{1-\omega}e^{\theta})^{\frac{p}{p-1}}}{(p-1)\cdot\left(\omega+(1-\omega)\cdot\left(\frac{\omega}{1-\omega}e^{\theta}\right)^{\frac{p}{p-1}}\right)^{1+\frac{1}{p}}} \end{cases} ,\ \theta\in(-\infty;+\infty) \tag{40}$$

It is remarkable that

$$e^{\theta} = -\frac{y'(\theta)}{x'(\theta)} \ . \tag{41}$$

It means that θ is a log-price parameter. Please compare (41) with (12).

Now let us apply an inverse 0.5-transform to (27) for a constant L. As a result we get a function

$$f(\theta) = \frac{L\left(\frac{\omega}{1-\omega}\right)^{\omega-\frac{1}{1-p}} e^{\frac{(p+1)\theta}{2(p-1)}}}{(1-p)\left(\omega+(1-\omega)\left(\frac{\omega e^{\theta}}{1-\omega}\right)^{\frac{p}{p-1}}\right)^{1+\frac{1}{p}}} . \tag{42}$$

Then using elementary transformations, let us omit their, we can get more compact and comprehensive formula

$$f(\theta) = \frac{L\cdot\left(\frac{\omega}{1-\omega}\right)^{\omega+\frac{1}{2p}}}{(1-p)(2\omega)^{\frac{p+1}{p}}}\left(\cosh\left(\frac{p}{2(1-p)}\left(\theta-\frac{\ln\frac{1-\omega}{\omega}}{p}\right)\right)\right)^{-\frac{p+1}{p}} . \tag{43}$$

Now it is clear that f has a maximum at a point $\theta_M = \dfrac{\ln\left(\frac{1-\omega}{\omega}\right)}{p}$ and f is an even with respect to θ_M.

Theorem 2. *For the function* $f(\theta)$ *from* (43) *the following equality holds true*

$$\int_{\mathbf{R}} f(t)dt = \frac{L\cdot 2^{1-\frac{1}{p}}\left(\dfrac{\omega}{1-\omega}\right)^{\omega+\frac{1}{2p}}\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)}{\omega^{1+\frac{1}{p}}\Gamma\left(\frac{1}{2p}\right)} , \tag{44}$$

Proof. Let us apply the integral $\int\limits_{-\infty}^{\infty}$ to both sides of (43). We get

$$\int_{-\infty}^{\infty} f(\theta)d\theta = \frac{L\cdot\left(\frac{\omega}{1-\omega}\right)^{\omega+\frac{1}{2p}}}{(1-p)(2\omega)^{\frac{p+1}{p}}}\cdot I , \tag{45}$$

where

$$I = \int_{-\infty}^{\infty}\left(\cosh\left(\frac{p}{2(1-p)}\left(\theta-\frac{\ln\frac{1-\omega}{\omega}}{p}\right)\right)\right)^{-\frac{p+1}{p}} d\theta . \tag{46}$$

In (46) we can make a change of variable $\tau = \frac{p}{2(1-p)}\left(\theta-\frac{\ln\frac{1-\omega}{\omega}}{p}\right)$ and get

$$I = \frac{2(1-p)}{p}\int_{-\infty}^{\infty}(\cosh(\tau)))^{-\frac{p+1}{p}} d\tau . \tag{47}$$

By definition of $\cosh\tau = \frac{e^{\tau}+e^{-\tau}}{2}$ we rewrite right hand side of (47) as

$$I = \frac{2(1-p)}{p}2^{\frac{p+1}{p}}\int_{-\infty}^{\infty}\left(e^{\tau}+e^{-\tau}\right)^{-\frac{p+1}{p}} d\tau , \tag{48}$$

and then

$$I = \frac{2^{2+\frac{1}{p}}(1-p)}{p} \int\limits_{-\infty}^{\infty} e^{(1+\frac{1}{p})\tau} \left(e^{2\tau}+1\right)^{-1-\frac{1}{p}} d\tau \ , \tag{49}$$

$$I = \frac{2^{2+\frac{1}{p}}(1-p)}{p} \int\limits_{-\infty}^{\infty} e^{\frac{1}{p}\tau} \left(e^{2\tau}+1\right)^{-1-\frac{1}{p}} de^{\tau} \ . \tag{50}$$

Next change of variable $t = e^{\tau}$ leads to

$$I = \frac{2^{2+\frac{1}{p}}(1-p)}{p} \int\limits_{0}^{\infty} t^{\frac{1}{p}} \left(t^2+1\right)^{-1-\frac{1}{p}} dt \ . \tag{51}$$

Finally, if we put $x = t^2$ in (51)

$$I = \frac{2^{2+\frac{1}{p}}(1-p)}{p} \int\limits_{0}^{\infty} x^{\frac{1}{2p}} \left(x+1\right)^{-1-\frac{1}{p}} 2^{-1} x^{-\frac{1}{2}} dx \ , \tag{52}$$

and after easy transformations we get

$$I = \frac{2^{1+\frac{1}{p}}(1-p)}{p} \int\limits_{0}^{\infty} \frac{x^{\frac{1}{2}(1+\frac{1}{p})-1}}{(x+1)^{1+\frac{1}{p}}} dx \ , \tag{53}$$

which can be expressed as the Euler's Beta function:

$$I = \frac{2^{1+\frac{1}{p}}(1-p)}{p} B\left(\frac{1}{2}\left(1+\frac{1}{p}\right), \frac{1}{2}\left(1+\frac{1}{p}\right)\right) \ . \tag{54}$$

Last expression with Beta function can be transformed via Euler's Gamma function:

$$I = \frac{2^{1+\frac{1}{p}}(1-p)}{p} \frac{\Gamma\left(\frac{1}{2}\left(1+\frac{1}{p}\right)\right) \cdot \Gamma\left(\frac{1}{2}\left(1+\frac{1}{p}\right)\right)}{\Gamma\left(1+\frac{1}{p}\right)} \ . \tag{55}$$

Furthermore we will apply Legendre duplication formula $\Gamma(z) \cdot \Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ to the numerator of (55) and $\Gamma(1+z) = z \cdot \Gamma(z)$ to the denominator of (55):

$$I = \frac{2^{1+\frac{1}{p}}(1-p)}{p} \frac{\Gamma\left(\frac{1}{2}\left(1+\frac{1}{p}\right)\right) \cdot 2^{1-\frac{1}{p}}\sqrt{\pi}\Gamma\left(\frac{1}{p}\right)}{\frac{1}{p}\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{2p}\right)} \ . \tag{56}$$

After reductions we get

$$I = \frac{2^2(1-p) \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{1}{2}\left(1+\frac{1}{p}\right)\right)}{\Gamma\left(\frac{1}{2p}\right)} \ . \tag{57}$$

Returning to (45) and substitute value of I into right hand side of (45) we get the statement of the theorem i.e. the equality (44). Q.E.D.

Now we can normalize the function $f(\theta)$ from (43) and can get a probability density function $\hat{f}$

$$\hat{f}(\theta) = \frac{f(\theta)}{\int\limits_{\mathbf{R}} f(t)dt} = \frac{\Gamma\left(\frac{1}{2p}\right)}{4(1-p)\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \cosh^{-1-\frac{1}{p}}\left(\frac{p}{2(1-p)}(\theta - \theta_M)\right) \ . \quad (58)$$

It is remarkable that mean value of a random variable distributed with $\hat{f}$ is θ_M and the variation of it depends only on p. The following theorem let us calculate the variation.

Lemma 1. *The next formula holds true*

$$\int\limits_0^{+\infty} \frac{t^{x-1}\ln^2 t dt}{(1+t)^{x+y}} = B''_{x^2}(x,y) - 2B''_{xy}(x,y) + B''_{y^2}(x,y) \ . \quad (59)$$

Proof. Let us take derivative from the integral $B(x,y) = \int\limits_0^{+\infty} \frac{t^{x-1}dt}{(1+t)^{x+y}}$ with respect to x:

$$\left(\int\limits_0^{+\infty} \frac{t^{x-1}dt}{(1+t)^{x+y}}\right)'_x = \int\limits_0^{+\infty} \frac{t^{x-1}\ln\frac{t}{1+t}dt}{(1+t)^{x+y}} \ . \quad (60)$$

Let us take derivative from the integral $\int\limits_0^{+\infty} \frac{t^{x-1}dt}{(1+t)^{x+y}}$ with respect to y:

$$\left(\int\limits_0^{+\infty} \frac{t^{x-1}dt}{(1+t)^{x+y}}\right)'_y = \int\limits_0^{+\infty} \frac{t^{x-1}\ln\frac{1}{1+t}dt}{(1+t)^{x+y}} \ . \quad (61)$$

Subtracting (61) from (60) we get

$$B'_x(x,y) - B'_y(x,y) = \int\limits_0^{+\infty} \frac{t^{x-1}\ln t dt}{(1+t)^{x+y}} \ . \quad (62)$$

Equation (62) can be considered as an application linear differential operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ to $B(x,y)$ in left hand side and as a result appearing a multiplier $\ln t$ in the integrand from the left hand side of the equation. If we again apply linear operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ to the left hand side of (62) we get

$$(B'_x(x,y) - B'_y(x,y))'_x - (B'_x(x,y) - B'_y(x,y))'_y = \int\limits_0^{+\infty} \frac{t^{x-1}\ln^2 t dt}{(1+t)^{x+y}} \ , \quad (63)$$

and after reduction we get

$$B'_{x^2}(x,y) - 2B'_{xy}(x,y) + B'_{y^2}(x,y) = \int\limits_0^{+\infty} \frac{t^{x-1}\ln^2 t dt}{(1+t)^{x+y}} \ . \tag{64}$$

Q.E.D.

Lemma 2. *The following formula is valid*

$$B'_{x^2}(x,y) - 2B'_{xy}(x,y) + B'_{y^2}(x,y) = B(x,y) \cdot \left((\psi(x)-\psi(y))^2 + \psi'(x) + \psi'(y)\right) , \tag{65}$$

where $\psi = \dfrac{\Gamma'}{\Gamma}$ *(it is called Polygamma function of zero order).*

Proof. Note that lhs of (65) is $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 B(x,y)$ and let us at first derive $B'_x(x,y)$ and $B'_y(x,y)$ differentiating with respect to x and y correspondingly the identity

$$\Gamma(x+y) \cdot B(x,y) = \Gamma(x) \cdot \Gamma(y) \ . \tag{66}$$

So,

$$\Gamma'(x+y) \cdot B(x,y) + \Gamma(x+y) \cdot B'_x(x,y) = \Gamma'(x) \cdot \Gamma(y) \tag{67}$$

leads to

$$\begin{aligned} B'_x(x,y) &= \frac{\Gamma'(x)\cdot\Gamma(y) - \Gamma'(x+y)\cdot B(x,y)}{\Gamma(x+y)} \\ &= B(x,y) \cdot \left(\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(x+y)}{\Gamma(x+y)}\right) \\ &= B(x,y) \cdot (\psi(x) - \psi(x+y)) \end{aligned} \tag{68}$$

and

$$\begin{aligned} B'_y(x,y) &= \frac{\Gamma(x)\cdot\Gamma'(y) - \Gamma'(x+y)\cdot B(x,y)}{\Gamma(x+y)} \\ &= B(x,y) \cdot \left(\frac{\Gamma'(y)}{\Gamma(y)} - \frac{\Gamma'(x+y)}{\Gamma(x+y)}\right) \\ &= B(x,y) \cdot (\psi(y) - \psi(x+y)) \ . \end{aligned} \tag{69}$$

Subtracting (69) from (68) gives us image of the linear differential operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$:

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) B(x,y) = B(x,y)(\psi(x) - \psi(y)) \ . \tag{70}$$

Applying the operator $\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ again to (70) we have

$$\begin{aligned} &\left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right)^2 B(x,y) \\ &= \left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right) B(x,y) \cdot (\psi(x)-\psi(y)) + B(x,y) \cdot \left(\tfrac{\partial}{\partial x} - \tfrac{\partial}{\partial y}\right)(\psi(x)-\psi(y)) \\ &= B(x,y) \cdot (\psi(x)-\psi(y))^2 + B(x,y) \cdot (\psi'(x) + \psi'(y)) \\ &= B(x,y) \cdot \left((\psi(x)-\psi(y))^2 + \psi'(x) + \psi'(y)\right) \ . \end{aligned} \tag{71}$$

Q.E.D.

Theorem 3. *The variation $\sigma^2_{\hat{f}}$ is equal to*

$$\frac{2(1-p)^2}{p^2} \cdot \psi'\left(\frac{1}{2}\left(1+\frac{1}{p}\right)\right), \text{ where } \psi = \frac{\Gamma'}{\Gamma} \ . \tag{72}$$

Proof. By definition $\sigma^2_{\hat{f}} = \int\limits_{-\infty}^{+\infty} \hat{f}(\theta) \cdot (\theta - \theta_M)^2 d\theta$ which can be reduced by change of variable $t = \theta - \theta_M$ to

$$\sigma^2_{\hat{f}} = \int\limits_{-\infty}^{+\infty} \frac{\Gamma\left(\frac{1}{2p}\right)}{4(1-p)\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \cosh^{-1-\frac{1}{p}}\left(\frac{pt}{2(1-p)}\right) \cdot t^2 dt \ . \tag{73}$$

The equality (73) we rewrite in a more convenient way:

$$\sigma^2_{\hat{f}} = \frac{2^{\frac{1}{p}-1}\Gamma\left(\frac{1}{2p}\right)}{(1-p)\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \int\limits_{-\infty}^{+\infty} \frac{e^{\frac{(1+p)t}{2(1-p)}} \cdot t^2 dt}{(1+e^{\frac{pt}{(1-p)}})^{1+\frac{1}{p}}} = \frac{2^{\frac{1}{p}-1}\Gamma\left(\frac{1}{2p}\right)}{(1-p)\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \cdot I \ . \tag{74}$$

The integral I can be rewritten as

$$I = \int\limits_{-\infty}^{+\infty} \frac{e^{\frac{(1+p)t}{2(1-p)}} \cdot t^2 dt}{(1+e^{\frac{pt}{(1-p)}})^{1+\frac{1}{p}}} = \left[\tau = e^{\frac{pt}{(1-p)}},\ t = \frac{(1-p)}{p}\ln\tau\right] =$$
$$= \frac{(1-p)^3}{p^3} \int\limits_{0}^{+\infty} \frac{\tau^{\frac{1-p}{2p}} \cdot \ln^2\tau d\tau}{(1+\tau)^{1+\frac{1}{p}}} \ . \tag{75}$$

By lemma 1 and lemma 2 we may change last integral in (75) with $B\left(\frac{1+p}{2p}, \frac{1+p}{2p}\right) \cdot \left(2\psi'\left(\frac{1+p}{2p}\right)\right)$ and as a result we continue (75) as

$$I = \frac{2(1-p)^3}{p^3} \cdot B\left(\frac{1+p}{2p}, \frac{1+p}{2p}\right) \cdot \psi'\left(\frac{1+p}{2p}\right) \ . \tag{76}$$

Finally, substituting value of I to (74) we get

$$\sigma^2_{\hat{f}} = \frac{2^{\frac{1}{p}-1}\Gamma\left(\frac{1}{2p}\right)}{(1-p)\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \cdot \frac{2(1-p)^3}{p^3} \cdot B\left(\frac{1+p}{2p}, \frac{1+p}{2p}\right) \cdot \psi'\left(\frac{1+p}{2p}\right) \ , \tag{77}$$

and after simplifying and reducing we get

$$\begin{aligned}\sigma_{\hat{f}}^2 &= \frac{2^{\frac{1}{p}}\Gamma\left(\frac{1}{2p}\right)}{\sqrt{\pi}\Gamma\left(\frac{1+p}{2p}\right)} \cdot \frac{(1-p)^2}{p^3} \cdot \frac{\Gamma\left(\frac{1+p}{2p}\right)\Gamma\left(\frac{1+p}{2p}\right)}{\Gamma\left(\frac{1+p}{p}\right)} \cdot \psi'\left(\frac{1+p}{2p}\right) \\ &= \frac{2^{\frac{1}{p}}}{\sqrt{\pi}} \cdot \frac{(1-p)^2}{p^3} \cdot \frac{\Gamma\left(\frac{1}{2p}\right) \cdot \Gamma\left(\frac{1+p}{2p}\right)}{\frac{1}{p}\Gamma\left(\frac{1}{p}\right)} \cdot \psi'\left(\frac{1+p}{2p}\right) \ .\end{aligned} \tag{78}$$

Using Legendre duplication formula $\Gamma(z) \cdot \Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ with $\Gamma\left(\frac{1}{2p}\right) \cdot \Gamma\left(\frac{1+p}{2p}\right)$ when $z = \frac{1}{2p}$ we will obtain

$$\sigma_{\hat{f}}^2 = \frac{2(1-p)^2}{p^2} \cdot \psi'\left(\frac{1+p}{2p}\right) \tag{79}$$

which is the statement of the theorem. Q.E.D.

For more convenient usage of the derived liquidity concentration we may approximate $\hat{f}$ by the Gaussian g having the expectation θ_M and a particular standard deviation σ_g

$$\sigma_g = \frac{\sigma_{\hat{f}}(1+1.15715p)}{0.99954+1.42359p}, \text{ where } \sigma_{\hat{f}}^2 = \int_{\mathbf{R}} \hat{f}(t) \cdot (t-\theta_M)^2 dt \ . \tag{80}$$

We have made the approximation with an accuracy 0.02 using a sup-norm on the segment $p \in [0.01; 0.99]$. Corresponding Gaussian g distinguishes from $\hat{f}$ less or equal than $\varepsilon(p)$ in sup-norm on a segment $\theta \in [-5\sigma; 5\sigma]$ i.e.

$$\max_{\theta \in [-5\sigma; 5\sigma]} |g(\theta) - \hat{f}(\theta)| \le \varepsilon(p) \ , \tag{81}$$

where $\varepsilon(p)$ is an increasing function ranging from 0.00002 at $p = 0.01$ to 1.127 at $p = 0.99$. Therefore, using standard deviation of $\hat{f}$ and parameter p we can find standard deviation for the Gaussian and vice versa knowing standard deviation of the Gaussian we can derive the standard deviation of $\hat{f}$. Here below we have graphs of a standard deviation from (72) and its inverse:

7.5 Wealth function for CES curve

Here we bound ourselves by a case $\omega = \frac{1}{2}$. Corresponding concentrated liquidity for CES trading curve (43) is

$$f(\theta) = \frac{L}{1-p} \cosh^{-1-\frac{1}{p}}\left(\frac{p(\theta-\theta_1)}{2(1-p)}\right), \tag{82}$$

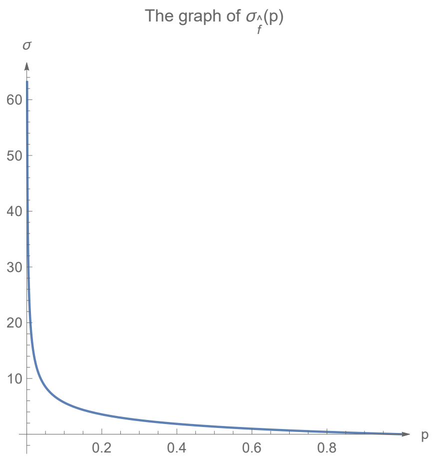

Fig. 19: The graph of $\sigma_{\hat{f}}(p)$.

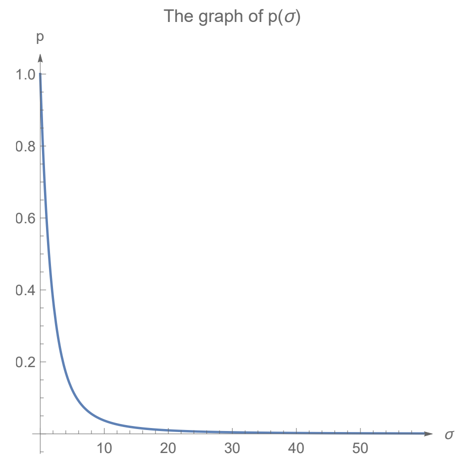

Fig. 20: The inverse of $\sigma_{\hat{f}}(p)$.

where θ_1 is a current log-price for a unit of resource noticed on the horizontal axis (for instance bitcoins are indicated on the horizontal axis while USDT are indicated on the vertical axis). Therefore, if we translate all liquidity concentration into the units of the vertical axis we get the wealth concentrated function $w(\theta, \theta_1)$:

$$w(\theta, \theta_1) := \begin{cases} f(\theta)e^{\frac{1}{2}\theta}, & \theta < \theta_1 \\ f(\theta)e^{-\frac{1}{2}\theta + \theta_1}, & \theta \geq \theta_1 \ . \end{cases} \tag{83}$$

In a first line stays an integrand from $-\infty$ to θ_1 from $\frac{1}{2}$-transform integral, see the formula (11) for $\omega = \frac{1}{2}$, which corresponds to the coordinate y in USDT tokens while the second line is an integrand corresponds to the x component multiplied on the rate e^{θ_1}. The multiplication has been made to express total wealth in the same units: USDT. So, when we integrate (83) we get the current total wealth for the log-rate θ_1:

$$S = y(\theta_1) + e^{\theta_1} \cdot x(\theta_1) \ . \tag{84}$$

For the sake of compactness we may rewrite (46) as

$$w(\theta, \theta_1) := e^{\frac{1}{2}\theta_1} f(\theta) e^{-\frac{1}{2}|\theta - \theta_1|}, \ \theta \in (-\infty, \ +\infty) \ . \tag{85}$$

Let us calculate S from (84). Hence,

$$
\begin{aligned}
S &= \int_{-\infty}^{+\infty} w(\theta,\theta_1)d\theta = \int_{-\infty}^{+\infty} e^{\frac{\theta_1}{2}} f(\theta) e^{-\frac{1}{2}|\theta-\theta_1|} d\theta \\
&= \frac{Le^{\frac{\theta_1}{2}}}{1-p} \int_{-\infty}^{+\infty} \cosh^{-1-\frac{1}{p}} \left(\frac{p(\theta-\theta_1)}{2(1-p)}\right) e^{-\frac{1}{2}|\theta-\theta_1|} d\theta \\
&= \frac{Le^{\frac{\theta_1}{2}}}{1-p} \int_{-\infty}^{+\infty} \cosh^{-1-\frac{1}{p}} \left(\frac{p\tau}{2(1-p)}\right) e^{-\frac{1}{2}|\tau|} d\tau \\
&= \frac{2Le^{\frac{\theta_1}{2}}}{1-p} \int_{0}^{+\infty} \cosh^{-1-\frac{1}{p}} \left(\frac{p\tau}{2(1-p)}\right) e^{-\frac{1}{2}\tau} d\tau \\
&= \frac{2Le^{\frac{\theta_1}{2}}}{1-p} \frac{2(1-p)}{p} \int_{0}^{+\infty} \cosh^{-1-\frac{1}{p}} (t)\, e^{-\frac{1-p}{p}t} dt \\
&= \frac{4Le^{\frac{\theta_1}{2}}}{p} 2^{1+\frac{1}{p}} \int_{0}^{+\infty} e^{(1+\frac{1}{p})t} \left(e^{2t}+1\right)^{-1-\frac{1}{p}} e^{(1-\frac{1}{p})t} dt \\
&= \frac{L2^{3+\frac{1}{p}} e^{\frac{\theta_1}{2}}}{p} \int_{0}^{+\infty} \left(e^{2t}+1\right)^{-1-\frac{1}{p}} e^{2t} dt \\
&= \frac{L2^{2+\frac{1}{p}} e^{\frac{\theta_1}{2}}}{p} \int_{0}^{+\infty} \left(e^{2t}+1\right)^{-1-\frac{1}{p}} de^{2t} \\
&= \frac{L2^{2+\frac{1}{p}} e^{\frac{\theta_1}{2}}}{p} \int_{1}^{+\infty} (u+1)^{-1-\frac{1}{p}} du \\
&= \frac{L2^{2+\frac{1}{p}} e^{\frac{\theta_1}{2}}}{p} (-p)\, (u+1)^{-\frac{1}{p}} \,|_1^{+\infty} = L2^{2+\frac{1}{p}} e^{\frac{\theta_1}{2}} \cdot 2^{-\frac{1}{p}} = 4Le^{\frac{\theta_1}{2}} \; .
\end{aligned}
$$

Finally, we can state

Theorem 4. *The probability distribution of the wealth is*

$$
\begin{aligned}
\hat{w}(\theta,\theta_1) &:= \frac{w(\theta,\theta_1)}{S} = \frac{e^{\frac{\theta_1}{2}} f(\theta) e^{-\frac{1}{2}|\theta-\theta_1|}}{4Le^{\frac{\theta_1}{2}}} \\
&= \frac{1}{4(1-p)} \cosh^{-1-\frac{1}{p}} \left(\frac{p(\theta-\theta_1)}{2(1-p)}\right) e^{-\frac{1}{2}|\theta-\theta_1|} \; .
\end{aligned} \tag{86}
$$

Theorem 5. *A first absolute momentum of the wealth distribution* $\hat{w}$ *is equal to*

$$A_{\hat{w}} = (1-p)\sum_{k=0}^{\infty} \frac{2^{-k}}{pk+1} \ . \tag{87}$$

Proof. By definition

$$A_{\hat{w}} = \int\limits_{-\infty}^{+\infty} \hat{w}(\theta, \theta_1) \cdot |\theta - \theta_1| d\theta \ . \tag{88}$$

We substitute the last expression from (86) to (88), change variable $\tau = \theta - \theta_1$, then utilizing evenness of integrand change $\int\limits_{-\infty}^{+\infty} = 2 \int\limits_{-\infty}^{+\infty}$, again change variable $t = \frac{p\tau}{2(1-p)}$ we get

$$A_{\hat{w}} = \frac{2(1-p)}{p^2} \int\limits_{0}^{+\infty} (\cosh(t))^{-1-\frac{1}{p}} e^{\left(1-\frac{1}{p}\right)t} t dt \ . \tag{89}$$

Further transformation of the right hand side (89) gives us a formula

$$\begin{aligned} A_{\hat{w}} &= \frac{2(1-p)}{p^2} \int\limits_{0}^{+\infty} \left(2^{-1} e^{-t}(e^{2t}+1)\right)^{-1-\frac{1}{p}} e^{\left(1-\frac{1}{p}\right)t} t dt \\ &= \frac{2^{2+\frac{1}{p}}(1-p)}{p^2} \int\limits_{0}^{+\infty} \left(e^{2t}+1\right)^{-1-\frac{1}{p}} e^{2t} t dt \ . \end{aligned} \tag{90}$$

Integrating by parts the last integral in (90) we will obtain

$$\int\limits_{0}^{+\infty} \left(e^{2t}+1\right)^{-1-\frac{1}{p}} e^{2t} t dt = \frac{p}{2} \int\limits_{0}^{+\infty} \left(e^{2t}+1\right)^{-\frac{1}{p}} dt \tag{91}$$

which after substitution to (90) leads us to

$$
\begin{aligned}
A_{\hat{w}} = \frac{2^{1+\frac{1}{p}}(1-p)}{p} \int\limits_0^{+\infty} \left(e^{2t}+1\right)^{-\frac{1}{p}} dt &= \begin{bmatrix} v = (e^{2t}+1)^{-1}, \\ dv = -2e^{2t}(e^{2t}+1)^{-2}dt, \\ \\ dv = -2v(1-v)dt, \\ dt = -\frac{1}{2}v^{-1}(1-v)^{-1}dv \end{bmatrix} \\
= \frac{2^{\frac{1}{p}}(1-p)}{p} \int\limits_0^{\frac{1}{2}} v^{\frac{1}{p}-1}(1-v)^{-1}dv &= \frac{2^{\frac{1}{p}}(1-p)}{p} \int\limits_0^{\frac{1}{2}} v^{\frac{1}{p}-1} \sum_{k=0}^{\infty} v^k dv \\
= \frac{2^{\frac{1}{p}}(1-p)}{p} \sum_{k=0}^{\infty} \int\limits_0^{\frac{1}{2}} v^{k-1+\frac{1}{p}} dv &= \frac{2^{\frac{1}{p}}(1-p)}{p} \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^{k+\frac{1}{p}}}{k+\frac{1}{p}} \\
&= (1-p) \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)^k}{pk+1} \ .
\end{aligned} \tag{92}
$$

Q.E.D.

Theorem 6. *A second momentum of the probability distribution of the wealth function equals to*

$$\sigma^2(\hat{w}) = \frac{2^{3+\frac{1}{p}}(1-p)^2}{p^2} \int\limits_0^{+\infty} (e^{2t}+1)^{-\frac{1}{p}} \cdot t dt \tag{93}$$

Proof. By definition

$$
\begin{aligned}
\sigma^2(\hat{w}) &= \int\limits_{-\infty}^{+\infty} \hat{w}(\theta, \theta_1) \cdot (\theta - \theta_1)^2 d\theta \\
&= \int\limits_{-\infty}^{+\infty} \frac{1}{4(1-p)} \cosh^{-1-\frac{1}{p}} \left(\frac{p \cdot (\theta - \theta_1)}{2(1-p)} \right) e^{-\frac{|\theta - \theta_1|}{2}} (\theta - \theta_1)^2 d\theta \\
&= \frac{1}{2(1-p)} \int\limits_{0}^{+\infty} \cosh^{-1-\frac{1}{p}} \left(\frac{p \cdot \tau}{2(1-p)} \right) e^{-\frac{\tau}{2}} \tau^2 d\tau \\
&= \frac{4(1-p)^2}{p^3} \int\limits_{0}^{+\infty} \cosh^{-1-\frac{1}{p}} (t)\, e^{\left(1-\frac{1}{p}\right)t} t^2 dt \\
&= \frac{4(1-p)^2}{p^3} \int\limits_{0}^{+\infty} \left(2^{-1} e^{-t} (e^{2t} + 1)\right)^{-1-\frac{1}{p}} e^{\left(1-\frac{1}{p}\right)t} t^2 dt \\
&= \frac{2^{3+\frac{1}{p}} (1-p)^2}{p^3} \int\limits_{0}^{+\infty} \left(e^{2t} + 1\right)^{-1-\frac{1}{p}} e^{2t} t^2 dt \\
&= \frac{2^{2+\frac{1}{p}} (1-p)^2}{p^3} \int\limits_{0}^{+\infty} \left(e^{2t} + 1\right)^{-1-\frac{1}{p}} t^2 de^{2t} \\
&= \frac{2^{2+\frac{1}{p}} (1-p)^2}{p^2} (-1) \int\limits_{0}^{+\infty} t^2 d\left(e^{2t} + 1\right)^{-\frac{1}{p}} \\
&= \frac{2^{2+\frac{1}{p}} (1-p)^2}{p^2} \left(-t^2 \cdot \left(e^{2t} + 1\right)^{-\frac{1}{p}} \Big|_{0}^{+\infty} + \int\limits_{0}^{+\infty} \left(e^{2t} + 1\right)^{-\frac{1}{p}} dt^2 \right) \\
&= \frac{2^{3+\frac{1}{p}} (1-p)^2}{p^2} \int\limits_{0}^{+\infty} \left(e^{2t} + 1\right)^{-\frac{1}{p}} t dt\ .
\end{aligned}
\tag{94}
$$

Q.E.D.

8 A Problem of CES Curve Preserving Constant Arbitrage Momentum

8.1 Description of the Problem

Let (X, Y) be a point in a coordinate system xOy where x denotes an amount of a base asset and y denotes an amount of a quote asset. Value X is an initial reserves of a base asset, Y is an initial reserves of a quote asset. Let ΔY be an arbitrage momentum — a part of the initial quote asset Y which DEX can offer to traders as a reward for their work on a recovery of a fair price. The fair price r can be obtained earlier from oracles. Traders are making transactions followed by CES protocol i.e. initial reserves (X, Y) are changing in accordance with the CES law:

$$\omega \cdot y^p + (1-\omega) \cdot x^p = \omega \cdot Y^p + (1-\omega) \cdot X^p \ , \tag{95}$$

where a parameter $p \in [0;\ 1]$ depends on price volatility and can be confessed by the oracles as well as the fair price r. The question is to find an equilibrium reserves (X_e, Y_e) i.e. the point belonging to curve (95) and price calculated at the point should be r. We recall that a price at a point on a curve is a minus one times a slope of a tangent curve passing through the point. Picture below can illustrate the problem.

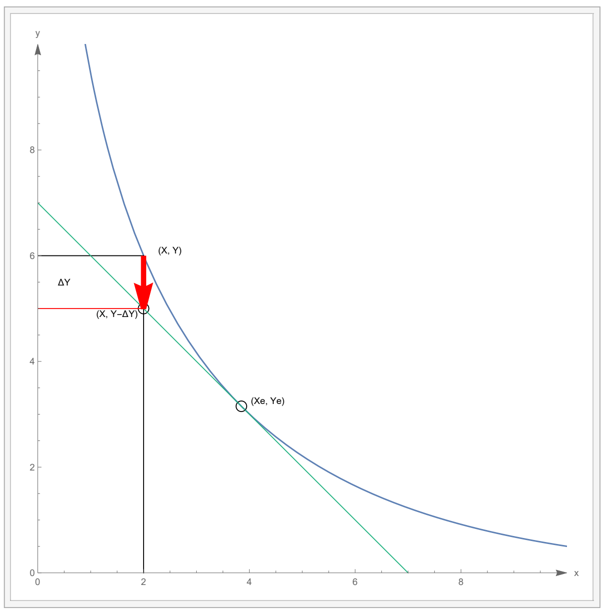

Fig. 21: CES curve preserving a quote asset arbitrage momentum ΔY

Note. *If traders will buy quote asset and sell base one in the state depicted on figure 21 the point $(X,\ Y)$ will move to $(X_e,\ Y_e)$. At some moment $(X,\ Y)$ will coincide to $(X_e,\ Y_e)$ but it might happen that $(X,\ Y)$ will pass through equilibrium $(X_e,\ Y_e)$ and will stop somewhere lower $(X_e,\ Y_e)$. Then the base asset arbitrage momentum will appear, see picture below.*

8.2 Deriving of main equations

Denote by $L^p = \omega \cdot Y^p + (1-\omega) \cdot X^p$ then (95) can be rewritten as

$$y = \left(\frac{L^p - (1-\omega)x^p}{\omega} \right)^{1/p} \ . \tag{96}$$

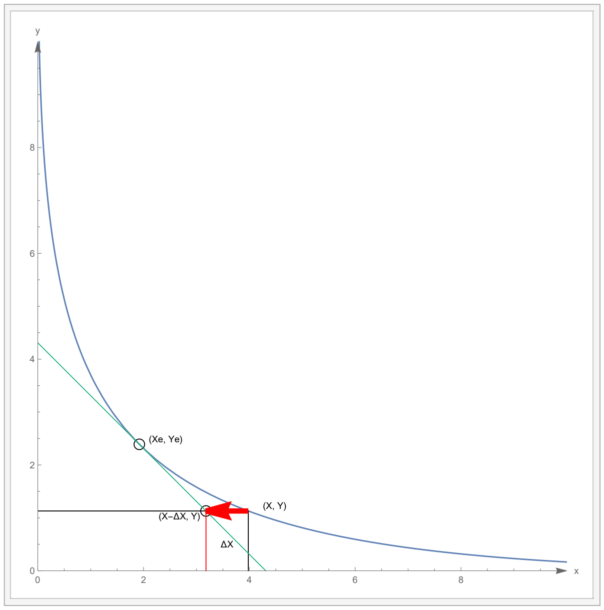

Fig. 22: CES curve preserving a base asset arbitrage momentum ΔX

Let us differentiate function y in (96) with respect to x:

$$y'_x = \frac{1}{p}\left(\frac{L^p - (1-\omega)x^p}{\omega}\right)^{1/p-1}\left(-\frac{1-\omega}{\omega}\right)\cdot p \cdot x^{p-1} \ . \tag{97}$$

Changing expression $\left(\frac{L^p-(1-\omega)x^p}{\omega}\right)^{1/p}$ in left hand side of (97) with y in accordance with (96) we get

$$y'_x = y^{1-p}\left(-\frac{1-\omega}{\omega}\right)\cdot x^{p-1} = \left(-\frac{1-\omega}{\omega}\right)\left(\frac{y}{x}\right)^{1-p} \ . \tag{98}$$

Taking into account $r = -y'_x$ at point (X_e, Y_e) we finally conclude that

$$r = \left(\frac{1-\omega}{\omega}\right)\left(\frac{y}{x}\right)^{1-p}|_{(X_e,Y_e)} = \left(\frac{1-\omega}{\omega}\right)\left(\frac{Y_e}{X_e}\right)^{1-p} \ . \tag{99}$$

This is a first equation. The second one comes directly from (95). Since the curve (95) contains (X_e, Y_e), therefore

$$\omega \cdot Y_e^p + (1-\omega)\cdot X_e^p = \omega \cdot Y^p + (1-\omega)\cdot X^p \ . \tag{100}$$

Since the fair price at the equilibrium point (X_e, Y_e) equals r a corresponding tangent line at this point has a form

$$y = -r(x - X_e) + Y_e \ . \tag{101}$$

From the other hand this line contains point $(X, Y - \Delta Y)$, therefore it gives true after substitution in (101)

$$Y - \Delta Y = -r(X - X_e) + Y_e \ . \tag{102}$$

Eventually, we have a system of equations (99),(100), and (102).

Note. For the case of a base asset arbitrage momentum depicted in 22 we should substitute in (101) $y = Y$, $x = X - \Delta X$, and get again (102) only we will recognize ΔY as $r \cdot \Delta X$

By elementary transformations one can reduce (99),(100) to the equation

$$a \cdot (c - X_e)^{1-p} + b \cdot X_e^{1-p} - c = 0, \tag{103}$$

where $a = \left(\frac{Y}{r}\right)^p$, $b = X^p$, $c = X + \frac{Y - \Delta Y}{r}$.

Note. *In the case of a base asset arbitrage momentum in parameter c we have to change $\Delta Y = r \cdot \Delta X$.*

If one solve (103) (for instance by Newton's method) the equilibrium quote asset Y_e immediately follows from (102), and from (99) one can get ω:

$$\omega = \left(\frac{r}{\left(\frac{Y_e}{X_e}\right)^{1-p}} + 1 \right)^{-1} . \tag{104}$$

So, we have to consider the equation (103) and solve it by Newton's method. For a fast and accurate solution of the equation we need to find a best initial value. But it is not whole story. After getting the equilibrium point (X_e, Y_e) our DEX can get a new input data: an updated r and p. Next subsection will discuss this point.

8.3 Initial value problem

Denote by G the left hand side of (103) then Newton's iterations will looks like

$$x_{n+1} = x_n - \frac{G(x_n)}{G'(x_n)}, \ x_0 =? \tag{105}$$

In theory if we guess x_0 quite close to the solution X_e the sequence $x_1, x_2, x_3, ...$ will tend to X_e. Denote by $q = 1 - p$. On figure 23 we depicts by a dashed blue curve the graph of $y = a(c - x)^q$ by a dashed orange curve the graph of $y = bx^q$, by a solid green curve the graph of their sum $y = a(c - x)^q + bx^q$ which is equivalent to $y = G(x) + c$ in our notations. So, the intersection of the green curve with black horizontal line $y = c$ will be the solution (X_e, Y_e). It is clear that a dashed red curve is a parallel to the blue one. It is obviously the graph of $y = a(c - x)^q + bc^q$ is the mentioned dashed red curve and the it's intersection point projection on x axis is a solution of $c = a(c - x)^q + bc^q$ which is

$$x_0 = c - \left(\frac{c - bc^q}{a} \right)^{\frac{1}{q}} . \tag{106}$$

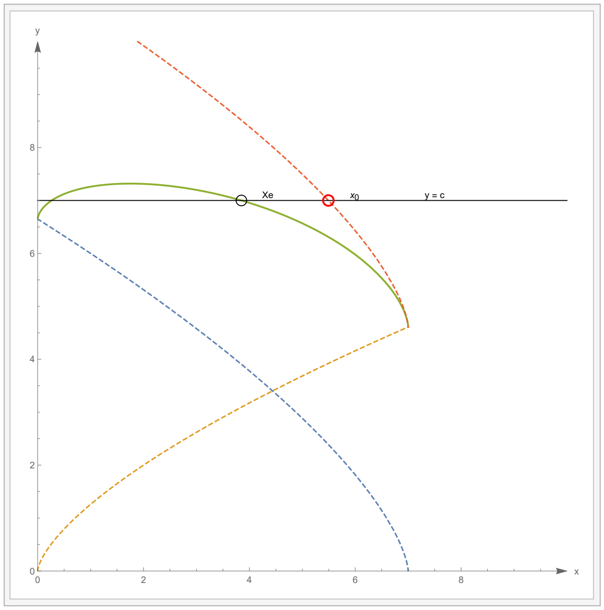

Fig. 23: Newton method initial value problem for a quote asset arbitrage momentum $\varDelta Y$

By construction one can see that $a(c-x)^q + bc^q > a(c-x)^q + bx^q$ for all $x \in [0;\ c]$ therefore $X_e < x_0 < c$. More over the function $y = G(x) + c$ is differential and convex in "up direction". That means sequence (105) initiated by x_0 from (106) tends to X_e with a rate prescribed by Newton's iteration method.

Combining figures 21 and 23 we get a figure describing a process of finding equilibrium point $(X_e,\ Y_e)$ for the quote asset arbitrage momentum $\varDelta Y$.

Note. *In the case of the base asset arbitrage momentum we have to consider function $y = bx^q$ and consider intersection point of a graph of $y = bx^q + a \cdot c^q$ and $y = c$, so in this case initial value x_0 has a form*

$$x_0 = \left(\frac{c - ac^q}{b}\right)^{\frac{1}{q}} . \tag{107}$$

Combining figures 22 and 25 we get a figure describing a process of finding equilibrium point $(X_e,\ Y_e)$ for the quote asset arbitrage momentum $\varDelta Y$

Ones we get X_e we calculate ω and restore CES curve passing through (X, Y) and (X_e, Y_e). With respect this curve traders will able to commit tradings obtaining some part of $\varDelta Y$ in reward and moving the current reserves from the state (X, Y) to the state (X_e, Y_e). And eventually points (X, Y) and (X_e, Y_e) will coincide and $\varDelta Y$ will vanish. Whether it happens or not but we may get an update of price r and parameter p. As a rule this changes are not big. If these changes are small we may quickly recalculate a new initial point as

$$x_0(r, p) = X_e(r_{old}, p_{old}) + (X_e)'_r dr + (X_e)'_p dp . \tag{108}$$

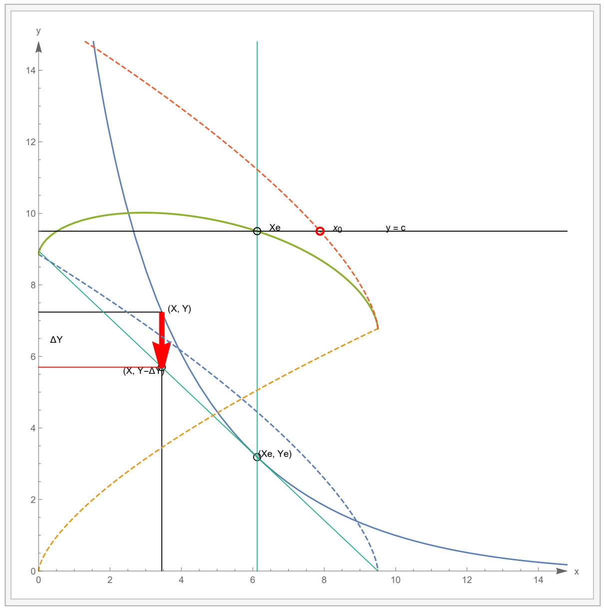

Fig. 24: Initial value and equilibrium point for a quote asset arbitrage momentum ΔY

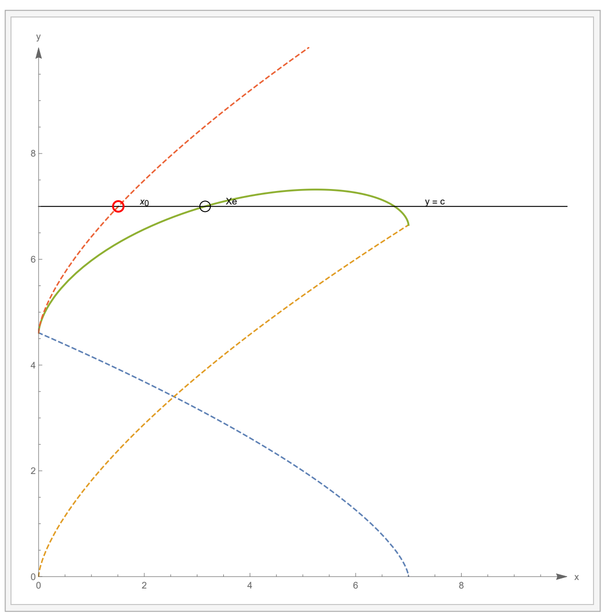

Fig. 25: Newton method initial value problem for a base asset arbitrage momentum ΔX

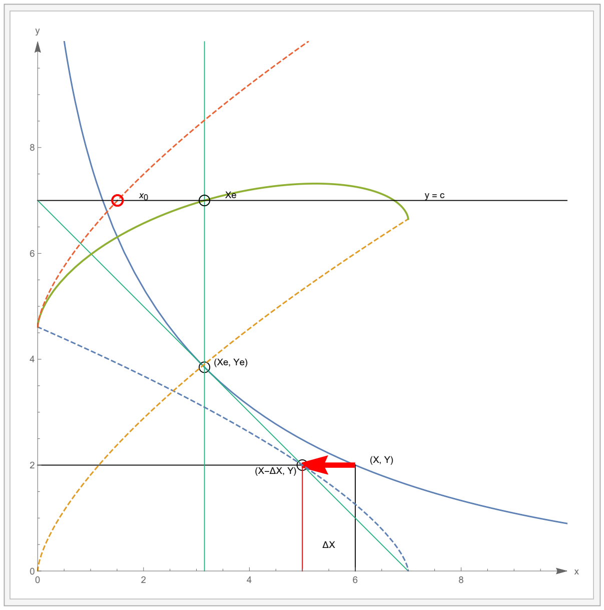

Fig. 26: Initial value and the equilibrium point $(X_e,\ Y_e)$ a base asset arbitrage momentum ΔX

We note that an expression in (108) after $X_e(r_{old}, p_{old})+$ is a differential of an implicit function $X_e(r,p)$ given by equation (103). Now we find partial derivatives from (108). For that we substitute $X_e = X_e(r,p)$ to (103) and get an identity

$$\begin{aligned} &a \cdot (c - X_e)^{1-p} + b \cdot X_e^{1-p} - c \equiv 0, \text{ where} \\ &a = \left(\frac{Y}{r}\right)^p, b = X^p, c = X + \frac{Y - \Delta Y}{r}. \end{aligned} \tag{109}$$

Taking into account that the coefficient $a,\ b,\ c$ are the functions in (r,p), p, r respectively we differentiate (109) with respect to r and then derive $(X_e)'_r$ from the obtained expression. After some simplifications we can get

$$(X_e)'_r = \frac{(X - c)\left(\frac{Y_e}{r}\right)^p}{-a\left(p\frac{Y_e}{r} + r(1-p)\right) + X^p(1-p)\left(\frac{Y_e}{rX_e}\right)^p}\ . \tag{110}$$

After change $\frac{Y_e}{r} = c - X_e$ it can be simplified to

$$(X_e)'_r = \frac{(X - c)\,(c - X_e)^p}{-a\left(p(c - X_e) + r(1-p)\right) + X^p(1-p)\left(\frac{c}{X_e} - 1\right)^p}\ . \tag{111}$$

The same technique can be applied to (109) but only with differentiating it with respect to p. After simplifications we will get

$$(X_e)'_p = \frac{a\,(c-X_e)^{1-p}\ln\left(\frac{Y}{r(c-X_e)}\right) + bX_e^{1-p}\ln\frac{X}{X_e}}{(1-p)\left(\frac{a}{(c-X_e)^p} - \frac{b}{X_e^p}\right)} \ . \tag{112}$$

Final simplifications leads us to the formula

$$(X_e)'_p = \frac{a\,(c-X_e)\cdot X_e^p\ln\left(\frac{Y}{r(c-X_e)}\right) + bX_e\cdot(c-X_e)^p\ln\frac{X}{X_e}}{(1-p)\left(aX_e^p - b\,(c-X_e)^p\right)} \ . \tag{113}$$

Resume. Resuming this subsection we have to say that if we denote initial value for new parameters (r, p) by x_0 and old parameters by (r_0, p_0) and denote by X_e an equilibrium base resource for old parameters then

$$x_0 = X_e + (X_e)'_r\cdot(r-r_0) + (X_e)'_p\cdot(p-p_0)\ , \tag{114}$$

and quantities $(X_e)'_r$ and $(X_e)'_p$ are calculated by (111) and (113).

9 Scaling coefficient for DEX's compare

Let us consider a following problem. Suppose we have two crypto exchanges — a small one and a big one both having the same types of base and quote assets only with different reserves of it. DEX_1 has a small amount of initial reserves $(X_1;\ Y_1)$ and DEX_2 has a big amount of initial reserves $(X_2;\ Y_2)$. Let $V(t)$ be a transaction volume density at the moment t. It can be the volume as for sell or for buy, so we may write $V = \max\{V_{\text{sell}},\ V_{\text{buy}}\}$. The question is how we may use transaction volume V on the big crypto currency exchange DEX_2 in order to estimate profitability DEX_1 using a scaling coefficient α?

To unswer the question we assume that there is a non-negative constant k have been derived statistically for any crypto currency. It shows trader's intolerance to currency rate changing: for instance $k = 0$ means trader's indifference to the currency rate slippage, it can be demonstrated in a stableswap model. The greater k the more intolerate traders to a rate changing. Now for any initial base and quote reserves $(X;\ Y)$ let us consider local coordinates originated at $(X;\ Y)$. We will call it as

$$\Delta X = x - X; \Delta Y = y - Y\ . \tag{115}$$

Therefore, we may formulate the problem of a profit scaling from DEX_2 to DEX_1 as

$$\Delta X = \alpha\cdot V\ , \tag{116}$$

and a coefficient α can be expressed via intolerance coefficient k by the formula

$$\alpha = e^{k\cdot\left(\ln\left(-\frac{\Delta Y}{\Delta X}\right)-\ln r\right)}\ , \tag{117}$$

where r is a fair rate of a token associated with axis ΔX in a units of token associated with axis ΔY and used out of DEX_1, and $-\frac{\Delta Y}{\Delta X}$ is an analogues rate on the DEX_1. This α is calculated when

$$\ln\left(-\frac{\Delta Y}{\Delta X}\right) \geq \ln r \ , \tag{118}$$

and shows seller's transaction volume (seller sells the base asset X) and we have a base asset growth on DEX_1:

$$\Delta X = \alpha V = e^{k\cdot\left(\ln\left(-\frac{\Delta Y}{\Delta X}\right)-\ln r\right)} V \ . \tag{119}$$

If $\ln\left(-\frac{\Delta Y}{\Delta X}\right) \leq \ln r$, then this case is profitable for a buyer (we have a quote asset growth), therefore

$$\Delta Y = \alpha V = e^{k\cdot\left(\ln r-\ln\left(-\frac{\Delta Y}{\Delta X}\right)\right)} V \ . \tag{120}$$

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